THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-ninth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, December 29–30, 1911, extending through three half-day sessions. The total attendance was fifty-seven, including the following forty-three members of the Society:

Professor F. Anderegg, Professor W. H. Bates, Professor G. A. Bliss, Dr. H. T. Burgess, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor Arnold Emch, Professor Peter Field, Dr. T. H. Hildebrandt, Dr. Louis Ingold, Professor Kurt Laves, Professor A. C. Lunn, Dr. W. D. MacMillan, Dr. H. F. MacNeish, Professor J. L. Markley, Professor William Marshall, Mr. R. M. Mathews, Professor Malcolm McNeil, Professor J. C. Morehead, Professor F. M. Morrison, Mr. E. J. Moulton, Professor F. R. Moulton, Professor Alexander Pell, Dr. Anna J. Pell, Professor W. S. Pemberton, Dr. R. E. Root, Mr. J. M. Rysgaard, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Professor J. B. Shaw, Mr. T. M. Simpson, Professor C. H. Sisam, Professor H. E. Slaught, Professor A. W. Smith, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Marion B. White, Professor E. J. Wilczynski, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor L. E. Dickson, Chairman of the Section, presided at the opening session on Friday morning. Professor E. B. Van Vleck presided at the session on Friday afternoon while Professor Dickson was delivering his address on the "History of the representations of numbers as the sum of squares," and Professor E. J. Townsend presided at the session on Saturday morning.

At the business meeting on Saturday morning the following officers of the Section for the year 1912 were elected: Professors D. R. Curtiss, chairman, H. E. Slaught, secretary, and A. L. Underhill, third member of the programme committee.

On Friday noon the members lunched together at the Quadrangle Club, and in the evening they dined together at the same place and spent one of the most enjoyable social occasions in the history of the Section.

The following papers were presented at this meeting:

(1) Professor ARNOLD EMCH: "Involutoric circular transformations as a particular case of the Steinerian transformation and their invariant net of cubics."

(2) Dr. R. E. ROOT: "Iterated limits in general analysis."

(3) Dr. ARNOLD DRESDEN: "Reduction of systems of linear differential equations of any order."

(4) Dr. LOUIS INGOLD: "Displacements in a function space."

(5) Professor L. E. DICKSON: "History of the representation of numbers as the sum of squares."

(6) Professor F. R. MOULTON: "Relations of families of periodic orbits in the restricted problem of three bodies."

(7) Professor L. E. DICKSON: "Note on Waring's theorem."

(8) Professor L. E. DICKSON: "Uniqueness of division in Cayley's algebras with eight units."

(9) Professor J. B. SHAW: "On differential invariants."

(10) Professor E. J. WILCZYNSKI: "On some geometric questions connected with the problem of three bodies."

(11) Professor PETER FIELD: "On Coulomb's laws of friction."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space."

(13) Mr. H. F. VANDIVER: "Theory of finite algebras."

(14) Dr. ARNOLD DRESDEN: "Note on the second variation; Jacobi's equation and Jacobi's theorem in the calculus of variations."

(15) Professor G. A. MILLER: "Gauss's lemma and some related group theory."

(16) Professor R. D. CARMICHAEL: "On a class of linear functional equations."

(17) Professor R. D. CARMICHAEL: "On the theory of the gamma function."

Mr. Vandiver's paper was communicated to the Society through Professor Dickson. In the absence of the authors, the papers of Mr. Vandiver, Dr. Bill, Professor Miller, and Professor Carmichael were read by title. Abstracts of the papers, except the historical paper by Professor Dickson, follow below in the order of the list above.

1. It is well known that with every Steinerian, or involutoric quadratic transformation in a plane is associated an invariant net of cubics passing through the fundamental quadruple of

the transformation and its diagonal points. Conversely, every plane cubic admits of an infinite number of birational transformations into itself, which form a continuous group. The involutoric circular transformation in a complex plane

$$z' = \frac{az + b}{cz - a}$$

is a particular case of a Steinerian transformation, and it is the purpose of Professor Emch's paper to establish this equivalence and the principal properties of the two nets of cubics which remain invariant in the transformations of the first and second kind, respectively.

2. The paper by Dr. Root is essentially an amplification, together with some applications, of a method for the investigation of iterated limits outlined in his paper at the April meeting of the Society. The method involves a generalization of the idea of neighborhood of an element, in the sense that a neighborhood of an element is a subclass of the range class specially related to the element. This relation of subclass to element is taken as undefined, and as the basis of the system of postulates. The character and form of the postulates are determined largely by two fundamental requirements: first, to provide for an adequate treatment of ideal limiting elements, and second, to insure the persistence of the specified conditions under composition of systems. The theorems on functions are, for the most part, direct analogues or generalizations of well-known theorems on multiple sequences and multiple and iterated limits of functions of real variables. Certain special features pertain, however, to the presence of ideal elements. The general theory is found to be available for functions on any range, subject to the Frechet notion of voisinage, or to properly conditioned K_1 and K_2 relations of the type used by E. H. Moore.

3. The purpose of Dr. Dresden's paper is to derive conditions under which a system of n ordinary linear differential equations in $n + 2$ variables and of order p may be reduced to a system of the same type in $n + 1$ variables.

If the given equations are of the form

$$\sum_{j=1}^{n+1} \sum_{l=0}^{p} a_{ilj} x_j^{(l)} = 0 \qquad (i = 1, \ldots, n), \tag{1}$$

where a_{ilj} are functions of t of class $C^{(l)}$ on a range $(t_1 t_2)$, we seek to determine conditions on the coefficients a_{ilj}, which will make it possible to write equations (1) in the form

$$\sum_{k=1}^{n} \sum_{l=0}^{p} A_{ik} w_k^{(l)} = 0 \qquad (i = 1, \ldots, n),$$

where

$$w_k = \sum_{j=1}^{n+1} \alpha_{kj} x_j.$$

The conditions arrived at are expressible as the vanishing of determinants of order $n+1$ in which the matrix a_{ipj} $(i = 1, \ldots, n; j = 1, \ldots, n+1)$ forms the first n columns, the last column being formed by sums of derivatives of various orders of the coefficients a_{ilj}.

4. In the first part of Dr. Ingold's paper a set (φ) of three mutually orthogonal unit vectors $\varphi_1, \varphi_2, \varphi_3$, is considered in relation to any set (ψ) of unit vectors ψ_1, ψ_2, ψ_3 (not necessarily orthogonal) into which the first can be transformed linearly. The vectors of both sets are regarded as functions of any number of parameters $u_1, u_2, \ldots$.

The projections of the derivatives $\partial\phi_i/\partial u_l$ upon the vectors φ_j are denoted by $P_{ij}^{(l)}$; similarly the projections of the derivatives $\partial\psi_i/\partial u_l$ upon the vectors ψ_j are denoted by $Q_{ij}^{(l)}$. A set of fundamental relations (R) is obtained connecting the quantities $P_{ij}^{(l)}$ with the $Q_{ij}^{(l)}$. A special case is the set of equations of Darboux* satisfied by the two-parameter rotations of a rigid system.

If the sets of vectors (φ), (ψ) contain only two vectors each, the corresponding relations (R) still hold and in a special case reduce to a single equation for the two-parameter rotations of a rigid plane tangent to a given surface.

In the second part of the paper the corresponding developments are obtained for a set of n mutually orthogonal normed functions of x $(\alpha \leqq x \leqq \beta)$ and any number of parameters, $u_1, u_2, \ldots$, say $\varphi_1(x; u_1, u_2, \ldots), \varphi_2(x; u_1, u_2, \ldots), \ldots, \varphi_n(x; u_1, u_2, \ldots)$. Quantities analogous to the $P_{ij}^{(l)}$, $Q_{ij}^{(l)}$ are obtained and a set of relations analogous to the relations (R) is found connecting them. Special cases of this set are generalizations of

* Darboux, Théorie générale des Surfaces, vol. 1, p. 49.

(1) The extensions to n dimensions and any number of parameters of Darboux's equations for the rotations of a rigid system.*
(2) Equations for the rotations of any rigid system moving tangent to any space.
(3) The Darboux extensions of the Lamé conditions for a triply orthogonal system of curves and surfaces in ordinary space.†

6. The theories of a number of classes of periodic solutions of the restricted problem of three bodies have been developed in papers presented to the Society by Professor Moulton in the last ten years. The present paper defines some additional periodic solutions consisting of orbits having multiple loops, examines the character of the changes through certain critical forms, and then takes up the problem of following out the analytic continuity, in the sense of analytic functions, of all the orbits with respect to the various parameters upon which the solutions depend, and of tracing the connections of the several families. Omitting reference to the nature of the branching and the definitions and qualifications necessary in some statements for complete precision, a few of the theorems on which the discussion depends are the following:

(1) Six orbits having real periods and single loops branch from one of zero dimensions at each of the finite masses, of which three are direct and three retrograde. In the vicinity of the origin only one each of the direct and of the retrograde orbits is real. They are symmetrical with respect to the x-axis.
(2) Six orbits having real periods and single loops branch from infinity, three of which are direct and three of which are retrograde with respect to fixed axes, while all are retrograde with respect to rotating axes. When the orbits are sufficiently large only one each of those which are direct and retrograde is real. They are symmetrical with respect to the x-axis.
(3) There is a geometrically single family of three-dimensional orbits branching from each of the five points of libration.

* These extensions are given by Hatzidakis, "Displacements depending on one, two, ..., k parameters in a space of n dimensions." *American Journal*, vol. 22 (1900), pp. 154–184.

† Darboux, Leçons sur les Systèmes orthogonaux et les Coordonnées curvilignes, Book II, Chapter I.

Those branching from the collinear libration points are symmetrical with respect to the xy and xz-planes; those branching from the equilateral triangular points are symmetrical only with respect to the xy-plane unless the finite bodies are equal, when they are symmetrical also with respect to the yz-plane.

(4) There is a geometrically single two-dimensional family branching from each of the collinear libration points in which the motion is retrograde with respect to the points.

(5) There are two-dimensional orbits associated with the equilateral triangular libration points falling under a number of cases depending upon the ratio of the finite masses and the parameter employed.

(6) No two finite loops can unite at only a single point.

(7) Single loops develop when, and only when, an orbit passes through a form of ejection or acquires a cusp on a surface of zero relative velocity.

(8) Multiple loops develop when, and only when, an orbit has a corner at a collinear libration point.

(9) If a direct orbit passes through a form of ejection or a cusp form it acquires a loop; a retrograde orbit under the same circumstances loses a loop.

(10) Ejection forms and cusp forms of periodic orbits can disappear only by uniting with other similar forms.

(11) Isolated periodic orbits do not exist.

(12) Three direct orbits have infinite periods; no retrograde orbits have infinite periods.

7. This paper by Professor Dickson gives an algebraic proof of the existence of an identity expressing any elementary symmetric function of $x_1^2, \ldots, x_e^2$ of degree $2m$ in the x's as a linear function with rational coefficients of M powers, with exponent $2m$, of linear functions of $x_1, \ldots, x_e$ with integral coefficients, where M and all the coefficients depend upon m and e alone.

8. In the second paper by Professor Dickson, it is pointed out that the algebras given by Cayley (*Philosophical Magazine*, volume 26 (1845), page 210; *American Journal of Mathematics*, volume 4 (1881), page 293) are equivalent to that with the units $1, e_1, \ldots, e_7$, where

$$e_i^2 = -1, \quad e_ie_j = -e_je_i. \qquad (i, j = 1, \ldots, 7;\ i \neq j),$$

$$e_1e_2 = e_3, \quad e_1e_4 = e_5, \quad e_1e_6 = e_7, \quad e_2e_5 = e_7,$$

$$e_3e_4 = e_7, \quad e_3e_5 = e_6, \quad e_4e_2 = e_6,$$

with 14 equations obtained by permuting the subscripts cyclically. It is now shown that right and left hand division, except by zero, is always possible and unique in this algebra. This is done by expressing the general element as a linear function of e_2, e_4, e_6, with coefficients B, C, ..., linear in e_1, and applying

$$e_jB = \bar{B}e_j, \quad (Be_j)(Ce_j) = (B\bar{C})e_j^2, \quad (Be_j)(Ce_k) = (\bar{B}\bar{C})(e_je_k),$$

for $j, k = 2, 4, 6$; $j \neq k$. Here $\bar{B} = r - se_1$ if $B = r + se_1$.

9. This is the third paper by Professor Shaw along this line, the other two having been presented before the Chicago Section at the meetings in December, 1910, and April, 1911. The method previously presented is now generalized to the case of space of n dimensions. It is shown in a suitable notation that the curvatures of a region of $n - m$ dimensions are expressible as the scalar invariants of the operator which is the generalization of the quaternion $S\,()\,\nabla, \sigma$. The various differential parameters are treated in the same manner. Finally the relation to the Maschke symbols is pointed out.

10. Two years ago Professor Wilczynski showed in a paper read before the Chicago Section how certain geometric problems connected with the problem of three bodies might be formulated, and how a number of these led to the question whether isosceles solutions exist. In the present paper this question finds a complete answer. If no two of the three masses are equal, the only isosceles solutions are the well-known equilateral solutions of Lagrange and its limiting cases. If two of the masses are equal, there exist two essentially distinct types of isosceles solutions.

11. Coulomb's laws of friction state that when one body slides over another the frictional force is (1) proportional to the normal pressure, (2) independent of the relative velocity, (3) independent of the area of the surfaces which are in contact. Painlevé in his Leçons sur le Frottement has shown that if

we assume these laws, peculiar situations may arise in case the problem is of such a nature that the normal pressure is a function of the coefficient of friction. Professor Field in this paper treats by graphical methods a problem of this kind which has been treated analytically by Painlevé and others. The paper will be published in the *Zeitschrift für Mathematik und Physik*.

12. This is the first of a series of papers, which Dr. Bill purposes to present to the Society giving the results obtained by applying to non-euclidean geometry the methods of analysis used by Professor Eduard Study for euclidean space. Starting from certain fundamental identities existing between the absolute invariants of the group of motions, the author arrives, in the first part of his paper, at a complete classification of the analytic curves of non-euclidean space. The second part of the paper is taken up with the derivation of the Frenet-Serret formulas for regular curves.

13. The theory of finite fields, due to Moore (Mathematical Papers, Chicago Congress, 1893, pages 208–242), may be regarded as a generalization of the arithmetical theory of congruences with prime moduli. In the present paper Mr. Vandiver examines an algebra which is an extension of the finite field, and which is defined as follows:

A set of s distinct elements forms a finite algebra of order S if the elements can be combined by addition, subtraction and multiplication, these operations being subject to the commutative and associative laws of elementary algebra, and if the resulting sum, difference, and product be each uniquely determined as an element of the set. Also there must exist in the set at least one element u_k such that the equation $u_k x = u_0$ (u_0 being the zero element) has the unique solution $x = u_0$.

In this algebra division is not always possible, and if possible, may give more than one quotient. Some of the elementary properties of a general finite algebra are given. Elements are classified as units and non-units. Prime and composite non-units are defined. It would appear that the theory includes as particular phases all arithmetical congruence theories, in particular, when the congruences considered have composite moduli.

14. In Dr. Dresden's note, a method is given for the discussion

of Weierstrass's treatment of the second variation, Jacobi's equation, and Jacobi's theorem in the calculus of variations for the case of an integral of the form $\int F(x, y, x', y')dt$. The method is based on the reduction of differential forms and unifies these three subjects; it is, moreover, immediately extensible to cases where the integrand contains more unknown functions, as an illustration of which the space problem is taken up.

15. According to Gauss's lemma any number m which is not divisible by the prime number p is a quadratic residue or a quadratic non-residue of p according as the series m, $2m$, $\cdots$, $\frac{1}{2}(p-1)m$ includes an even or an odd number of numbers whose least absolute residues (mod p) are negative. The main objects of Professor Miller's paper are to exhibit the setting of this lemma in the theory of abelian groups and to show how readily it can be deduced from this theory.

A set of operators of a group G is called a complete set for the nth powers if the nth powers of these operators give all the different nth powers of the operators of G and if no two operators of the set have the same nth power. The products obtained by multiplying all the operators of a complete set for the nth powers of an abelian group by any operator of the group constitute a complete set for the nth powers. The h complete sets obtained by multiplying any one such complete set by the different operators of the abelian group whose orders divide n are called complementary sets. These complementary sets involve every operator of the abelian group once and only once. If all the numbers in any complete set for squares (mod p) of the numbers 1, 2, $\cdots$, $p-1$ are multiplied by any one α of these numbers, then α is a quadratic residue or a quadratic non-residue of p according as an even or an odd number of these products occur in the set which is complementary to the given set for squares. This theorem clearly includes Gauss's lemma.

16. It is a problem of great difficulty (and one which is not completely solved in any extensive class of cases) to determine the general solution of a functional equation which involves two operations, as in the case of partial differential equations or mixed equations. The object of the present paper is to investigate the properties of the solutions of an equation belonging

to this general class; namely, an equation of the form

$$f(x+\omega_1+\omega_2) + \varphi(x)f(x+\omega_1) + \psi(x)f(x+\omega_2) + \chi(x)f(x) = 0,$$

certain restrictions being placed on the coefficients $\varphi(x)$, $\psi(x)$, $\chi(x)$. Such an equation involves the two operations of replacing x by $x+\omega_1$ and by $x+\omega_2$. The general theory of these equations throws light on the nature of the solutions of various functional equations involving two operations.

Among the results which Professor Carmichael obtains are the following: The arbitrary elements in the solution of the equation are doubly periodic functions of periods ω_1 and ω_2. Regions in which the solutions may be chosen arbitrarily are found; and by means of such regions it is shown that an infinite number of arbitrary elements must enter into a general solution of the equation. Analytic solutions involving simply periodic functions (some of period ω_1 and others of period ω_2) are obtained. A means is also given for expressing linearly in terms of particular solutions of this type (the multipliers being doubly periodic functions) any one of a large class of quasi-general solutions.

17. From the general theory of difference equations (see papers by Carmichael and by Birkhoff in the *Transactions*, volume 12) it follows that the equation $g(x+1) = xg(x)$ has two solutions $g_1(x)$ and $\bar{g}_1(x)$ having the property that each of the limits

$$\lim_{x=+\infty} g_1(x)x^{-x+\frac{1}{2}}e^x \quad \text{and} \quad \lim_{x=-\infty} x^{-x+\frac{1}{2}}e^x$$

exists and is 1, where x goes to infinity along the axis of reals in the positive sense in the first case and in the negative sense in the second case. Moreover,

$$\bar{g}_1(x) = (1 - e^{2\pi x\sqrt{-1}})g_1(x).$$

Also, any solution $g(x)$ is uniquely determined if $g(1)$ is assigned (different from zero) and

$$\lim_{x=+\infty} g(x)x^{-x+\frac{1}{2}}e^x$$

exists. It is natural to choose for the definition of $\Gamma(x)$ one that associates it most intimately with this general theory. Accordingly, Professor Carmichael defines a first and a second gamma function, $\Gamma(x)$ and $\bar{\Gamma}(x)$, as those solutions of the

equation $g(x+1) = xg(x)$ which have the properties that

$$\Gamma(1) = 1, \qquad \bar{\Gamma}(x) = (1 - e^{2\pi x\sqrt{-1}})\Gamma(x),$$

and

$$\lim_{x=+\infty} \Gamma(x)x^{-x+\frac{1}{2}}e^{x} \text{ exists.}$$

If one starts from these definitions and makes use of the general theory of linear homogeneous difference equations of the first order, the fundamental properties of $\Gamma(x)$ and $\bar{\Gamma}(x)$ are readily obtained. The theory, as worked out recently by Professor Carmichael in his academic lectures, is decidedly simpler and more elegant than the usual theory of the gamma function, as developed, for instance, in Nielsen's Handbuch.

H. E. SLAUGHT,
Secretary of the Section.

AN IDENTICAL TRANSFORMATION OF THE ELLIPTIC ELEMENT IN THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society, April 29, 1911.)

THIS paper is based upon a formula published in 1865 in a pamphlet entitled "Problemata quaedam mechanica functionum ellipticarum ope soluta.—Dissertatio inauguralis," by G. G. A. Biermann (Berolini), where it is quoted as derived from Weierstrass's lectures. The formula is, after correcting slight misprints in Biermann's pamphlet,

$$(1) \quad F(x) = x_0 + \frac{\sqrt{R(x_0)}\sqrt{S} + \frac{1}{2}R'(x_0)[s - \frac{1}{24}R''(x_0)] + \frac{1}{24}R(x_0)R'''(x_0)}{2[s - \frac{1}{24}R''(x_0)]^2 - \frac{1}{2}A \cdot R(x_0)}.$$

F is the solution of

$$(2) \quad (F')^2 = AF^4 + 4BF^3 + 6CF^2 + 4B'F + A' = R(F).$$

The accents used with F and R denote differentiation, x_0 is an arbitrary constant, and A, B, C, B', A' are constant

coefficients. Also

$$S = 4s^3 - g_2 s - g_3 = 4(s - \epsilon_1)(s - \epsilon_2)(s - \epsilon_3),$$

$$(3) \qquad s = \wp(x), \quad g_2 = AA' + 3C^2 - 4BB',$$

$$g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3.$$

In Enneper, Elliptische Functionen (Halle, 1890), pages 27–30, may be found considerable discussion of the preceding formulas, while on page 59 another form of (1) is used in obtaining the addition theorem for $\wp(u)$.

Greenhill, Elliptic Functions (London, 1892), page 151, mentions (1) in connection with the reduction of the general elliptic element $dx/\sqrt{X}$ to Weierstrass's canonical form $ds/\sqrt{S}$. Haentzschel's use of Weierstrass's formula will be considered later.

From (1) an identical transformation of $ds/\sqrt{S}$ may be obtained as follows: By a proper choice of constants $\alpha, \beta, \gamma, \delta$, the linear transformation

$$(4) \qquad F = \frac{\alpha v + \beta}{\gamma v + \delta}$$

changes (2) into

$$(5) \qquad (v')^2 = 4v^3 - g_2 v - g_3,$$

whence, writing

$$(6) \qquad \bar{F} - x_0 = v,$$

(5) becomes

$$(7) \qquad (\bar{F}')^2 = 4(\bar{F} - x_0)^3 - g_2(\bar{F} - x_0) - g_3 = \bar{R}(\bar{F}).$$

Hence

$$\bar{R}(x_0) = -g_3, \quad \bar{R}'(x_0) = -g_2,$$

$$(8) \qquad \bar{R}''(x_0) = 0, \quad \bar{R}'''(x_0) = 4! \quad \bar{R}''''(x_0) = 0.$$

Then using the form $\bar{R}$ as indicated in (7) for R in the fundamental formula (1) gives

$$(9) \qquad v = \frac{\sqrt{(-g_3)}\sqrt{(4s^3 - g_2 s - g_3)} - \frac{1}{2}g_2 s - g_3}{2s^2}.$$

When (9) is rationalized and simplified by writing $g_2 = 4a$, $g_3 = 2b$, it gives

$$(10) \qquad v^2 s^2 + 2avs + 2b(v + s) + a^2 = 0.$$

Referring to (5), it appears that the general biquadratic X or $R(F)$ has now, by the changes above, become the same as in Weierstrass's elliptic element, so that the transformation (10) changes $ds/\sqrt{S}$ into itself, i. e., is an identical transformation, a fact which may be easily verified.

Equation (10) is obviously symmetrical in v and s, while (9) shows that the transformation is irrational in both directions. The curve corresponding to (10) has no node, except when $g_2^3 = 27g_3^2$, but consists of two distinct branches, asymptotic to both axes. Evidently the line $v - s = 0$ is an axis of symmetry and it may be shown to intersect the curve twice or not at all, according as $g_2^3 - 27g_3^2$ is negative or positive, this expression being also the criterion for the two types of $\wp$-function curves, the basis of Haentzschel's discussion.

Haentzschel, in his Reduction der Potentialgleichung (Berlin, 1893), wished to obtain a pair of families of orthogonal curves by the aid of conjugate functions. To this end he wrote

$$s = \wp(t + iu), \quad \sigma = \wp(t - iu), \tag{11}$$

$$x + iy = F(t + iu), \quad x - iy = F_1(t - iu), \tag{12}$$

in which F and its conjugate F_1 were defined by (2), and x_0 was complex. The orthogonal families came from the elimination of u or t respectively from (12) and writing either ρ or ρ_1 as a parameter, where

$$\rho = \wp(2t) \quad \rho_1 = \wp(2iu). \tag{13}$$

He treated in detail the cases in which x_0 is any one of the roots of $R(x) = 0$, equation (2).

In the *Archiv der Mathematik und Physik*, series 3, volume 10, Hefte 3/4, pages 234–237, the writer has discussed some of Haentzschel's results and referred to other articles on the same point. On pages 33–37 (Potentialgleichung), Haentzschel treated the general case, i. e., x_0 unrestricted. This depends upon the $\wp$-function curves obtained from

$$\begin{aligned} x + iy &= s = \wp(t + iu), \\ x - iy &= \sigma = \wp(t - iu) \end{aligned} \tag{14}$$

[ρ and ρ_1 as in (13), $a = g_2/4$, $b = g_3/2$], viz.,

$$\begin{aligned} (15)\quad & s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho \\ & + s\sigma(2a - 2\rho^2) + (s + \sigma)(2\rho a + 2b) = 0. \end{aligned}$$

Haentzschel's work upon this topic concluded with an outline of the elimination of s and σ between (15) and two equations analogous to (10).

In the BULLETIN for June, 1899, pages 431–437, the writer has discussed the case in which

$$R(F) = A[\pm(F - x_0)^4 \pm c(F - x_0)^2 \pm 1], \tag{16}$$

and has factored the result, obtaining four curves each of the fourth degree.

The remainder of this paper is to be devoted to the case in which

$$\bar{R}(\bar{F}) = 4(\bar{F} - x_0)^3 - g_2(\bar{F} - x_0) - g_3. \tag{7}$$

First s and σ are to be eliminated from (15) and (10) together with the conjugate of the latter, i. e., from

$$v^2s^2 + 2avs + 2b(v + s) + a^2 = 0, \tag{17}$$

$$w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0, \tag{18}$$

$$[v = x + iy, \quad w = x - iy]. \tag{19}$$

In conclusion the resulting equation will be resolved into three components, of which two are the original $\wp$-function curves. Following Haentzschel, radicals may be avoided by writing (17) thus:

$$s^2 = -\frac{2s(av + b) + 2bv + a^2}{v^2}, \tag{20}$$

with a similar value for σ^2.

Substituting these values of s^2 and σ^2 in (15) gives

$$Ds\sigma + Ps + Q\sigma + E = 0, \tag{21}$$

$$\begin{aligned}
\tfrac{1}{2}D &= av^2w^2 - \rho^2v^2w^2 + 2\rho av^2w + 2\rho avw^2 + 2a^2vw + 2\rho bv^2 \\
&\quad + 2\rho bw^2 + 2abv + 2abw + 2b^2, \\
\tfrac{1}{2}P &= \rho av^2w^2 + bv^2w^2 + 2\rho bv^2w - \rho^2avw^2 + 2abvw + \rho a^2v^2 \\
&\quad - \rho^2bw^2 + a^3v + 2b^2w + a^2b, \\
\tfrac{1}{2}Q &= \rho av^2w^2 + bv^2w^2 + 2\rho bvw^2 - \rho^2av^2w + 2abvw + \rho a^2w \\
&\quad - \rho^2bv^2 + a^3w + 2b^2v + a^2b, \\
E &= 2\rho bv^2w^2 + a^2v^2w^2 - 2\rho^2bvw^2 - 2\rho^2bv^2w + 4b^2vw - \rho^2a^2v^2 \\
&\quad - \rho^2a^2w^2 + 2a^2bv + 2a^2bw + a^4.
\end{aligned} \tag{22}$$

Solving (21) for s and substituting in (17) gives a quadratic in σ, between which and (18), another quadratic, σ is to be eliminated. The result is

$$(23)\quad L^2 - b^2(PQ - DE)^2(4v^3 - 4av - 2b)(4w^3 - 4aw - 2b) = 0,$$

$$\begin{aligned}(24)\quad L = {} & v^2w^2E^2 - 2(PQ - DE)(av + b)(aw + b) \\ & + Pw^2[P(a^2 + 2bv) - 2E(av + b)] + Qv^2[Q(a^2 + 2bw) \\ & - 2E(aw + b)] + [D(a^2 + 2bv) - 2Q(av + b)][D(a^2 \\ & + 2bw) - 2P(aw + b)].\end{aligned}$$

As would be expected, (21), (22), and (23) are symmetrical in v and w, while $(v - \epsilon_1)(v - \epsilon_2)(v - \epsilon_3)(w - \epsilon_1)(w - \epsilon_2)(w - \epsilon_3)$ from the last two factors of (23) may be used to find the foci of the curves represented. Each term in (24), on reduction, is found to contain the factor v^2w^2, and this being true of $PQ - DE$ which occurs also in (23), it is possible to cancel the factor v^4w^4 from the latter and at last obtain in v and w coordinates the equation of either of the pair of orthogonal curves. The factoring and consequent resolution of (23) is much facilitated by first treating the simple case for which $\rho = 0$. The degenerate form of (15) is

$$(25)\quad s^2\sigma^2 + 2as\sigma + 2b(s + \sigma) + a^2 = 0,$$

which is of the same type as (17) and (18). Elimination of s and σ from these three gives

$$(26)\quad [v^2w^2 + 2avw + 2b(v + w) + a^2]^3[\varphi(v, w)] = 0.$$

The last factor of (26) is the form which (15) becomes when $\rho = -4(2b^2 - a^3)/a^4$, $s = v$, $\sigma = w$. But the triple factor in (26) is not preserved when the general case $\rho \neq 0$ is treated. However (15), which it will be remembered is the equation of the $\wp$-function curves, has the remarkable property of being symmetrical in all *three* of the quantities ρ, s and σ.

By reasoning which will presently be given, it may be inferred that one factor of (23) is the first member of

$$\begin{aligned}(27)\quad & v^2w^2 + 2\rho b + a^2 + (v^2 + w^2)\rho^2 - vw(v + w)2\rho \\ & + vw(2a - 2\rho^2) + (v + w)(2\rho a + 2b) = 0.\end{aligned}$$

Now (23) is the result as stated of eliminating s and σ from the following:

$$s^2\sigma^2 + 2\rho b + a^2 + (s^2 + \sigma^2)\rho^2 - s\sigma(s + \sigma)2\rho + s\sigma(2a - 2\rho^2) + (s + \rho)(2\rho a + 2b) = 0, \tag{15}$$

$$v^2s^2 + 2avs + 2b(v + s) + a^2 = 0, \tag{17}$$

$$w^2\sigma^2 + 2aw\sigma + 2b(w + \sigma) + a^2 = 0. \tag{18}$$

Elimination of s from (15) and (17) gives an equation symmetrical in v and σ, which is thus identical with the result of eliminating w from (27) and (18), so that (27) is consistent with (15), (17) and (18). It may be shown by actual division that the *square* of (27) is a factor of (23). In carrying out the division, the symmetry of (27) as to ρ, v and w, in connection with the assumed symmetry of (23) in the same quantities, permits many terms of the quotient to be written by inspection after a few have been obtained. At this stage the knowledge of the factors for the special case $\rho = 0$ determines many coefficients in the general case. The entire computation has been carefully checked, and the single assumption of symmetry verified, but the length of the quotient (one hundred ninety-eight terms) prevents its reproduction in this paper. It is of particular importance that this identical transformation, when employed to obtain new curves from the $\wp$-function curves in connection with the theory of conjugate functions, reproduces the original curves.

UNIVERSITY OF PENNSYLVANIA,
April, 1911.

SURFACES IN HYPERSPACE WHICH HAVE A TANGENT LINE WITH THREE-POINT CONTACT PASSING THROUGH EACH POINT.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, December 27, 1911.)

THROUGH each point on a surface in ordinary space S_3 pass two tangents having with the surface three-point contact (tangents to the asymptotic lines). The osculating planes to these curves are also tangent to the surface at the point of osculation. It is easily seen that the lines on a ruled surface in hyperspace have these same properties. The question

then naturally arises whether there exist in hyperspace surfaces not ruled having the same sort of lines. It is the object of this paper to show the existence of such surfaces and to derive the differential equation which the coordinates must satisfy. The results are generalized for spreads of higher dimensions than two.

1. The existence of such surfaces can be shown geometrically as follows: On a ruled surface in a space S_n trace three infinitely near curves c, c_1, c_2. Through the curves c_1 and c_2 pass a second ruled surface infinitely near the first and on this surface trace a third curve c_3 infinitely near to c_1, c_2. Then through the curves c_2, c_3 pass a third ruled surface infinitely near to the second and on it trace a curve c_4 infinitely near c_2, c_3. Continuing this process to the limit, it is at once evident that the locus of the curves c is a surface of the kind desired.

We will now derive the condition that a surface in S_n shall have through each point a curve whose osculating plane is tangent to the surface.

Let the surface be

$$x^{(i)} = x^{(i)}(u, v) \quad (i = 1, 2, 3, \cdots, n+1) \tag{1}$$

and let three neighboring points be

$$x, \quad (2) \qquad x_1 du + x_2 dv, \quad (3)$$*

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + x_1 d^2u + x_2 d^2v. \tag{4}$$

The tangent plane to the surface at the point x is defined by the three points†

$$x, \ x_1, \ x_2.$$

If the given curve is to have this plane for osculating plane, it is only necessary that the point

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 \tag{5}$$

should lie in it; for the points (2), (3), and $x_1 d^2u + x_2 d^2v$ already lie in it. The condition then that (4) should lie in the tangent plane is

$$x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + Ax_1 + Bx_2 + Cx = 0. \tag{6}$$

* Subscripts denote derivatives with respect to a variable t, where the curve on the surface is defined by assuming u and v to be functions of t. Hereafter superscripts will be omitted where no ambiguity occurs.

† See Segre, "Su una classa di superficie degl'iperspazii ecc.", *Atti di Torino*, 1907.

If the coordinates satisfy (6), then the points (2), (3), (4) will lie in the tangent plane. Hence:

If the coordinates of a surface in S_n satisfy a parabolic partial differential equation of the second order, then through each point of the surface passes a curve whose osculating plane coincides with the tangent plane to the surface at the point, and conversely.*

If now a tangent line is to have three-point contact with the surface, two conditions are necessary: (*a*) the point (5) must lie in the tangent plane; (*b*) for some choice of $d^2u : d^2v$ the point (4) must lie on the tangent line. The point (4) lies on the line joining (5) to $x_1d^2u + x_2d^2v$, and consequently if (6) is satisfied then when $d^2u : d^2v$ varies the point (4) will describe a line in the tangent plane. This line will cut the tangent line drawn in the direction du/dv. Thus if condition (*a*) is satisfied, condition (*b*) is also satisfied. Hence:

The necessary and sufficient condition that a surface in S_n have through each point a tangent line having three-point contact with the surface is that the coordinates of the surface satisfy a parabolic partial differential equation of the second order.

From the form of equation (6) it is seen that the directions of the three-point tangents are in the direction of the characteristics, since each is in the direction defined by du/dv.

In the paper referred to above Segre showed that a surface whose coordinates are particular solutions of two general partial differential equations of the second order must either be a developable or else lie in an ordinary space of three dimensions. From the above property it is evident that the surface could not be a developable if the coordinates satisfy two parabolic differential equations, for that would necessitate two principal directions (directions of three-point contact), which would be impossible. Then if such be the case, the surface must lie in an ordinary space S_3. In the pencil of partial differential equations which is defined by two general ones there are always two parabolic ones. Hence the two partial differential equations which the coordinates of a developable surface in hyperspace satisfy cannot be independent but must be such that there is only one parabolic equation belonging to the pencil defined by them.

2. The above considerations can be extended to varieties

* See Segre, loc. cit., §15.

of any number of dimensions contained in a space S_n of n dimensions. Let the coordinates of the variety V_m be

$$x^{(i)} = x^{(i)}(u_1 u_2 \cdots u_m) \quad (i = 1, 2, \cdots, n+1). \tag{7}$$

The tangent S_m to V_m is defined by the points

$$x, \quad x_1, \quad x_2, \quad \cdots, \quad x_m$$

and if the osculating plane to a curve traced on the variety lies in this tangent S_m, then the three points

$$(8)\ x, \qquad (9)\ \sum_{i=1}^{i=m} x_i du_i,$$

$$\sum_{i=1}^{i=m}\sum_{j=1}^{j=m} x_{ij} du_i du_j + \sum_{i=1}^{m} x_i d^2 u_i \tag{10}$$

must lie in it. But as before, (8), (9), and $\sum_{i=1}^{m} x_i du_i$ already lie in the tangent S_m and therefore the only condition necessary is that the point

$$\Sigma\Sigma x_{ij} du_i du_j \tag{11}$$

should lie in it. This condition is

$$\sum_{i,j=1}^{m} x_{ij} du_i du_j + \sum_{i=1}^{m} A_i x_i + A_{n+1} x = 0. \tag{12}$$

It is evident, since there are $m-1$ independent quantities $du_1 : du_2 : du_3 : \cdots : du_m$ at our disposal and $m+1$ quantities A, that equation (12) can always be satisfied if $2m > n$. And if $2m < n$, then the coordinates x must be particular solutions of (12) if the osculating plane to a curve traced on V_m lies in the tangent S_m. In this case, as in the preceding one, it is seen that if the point (11) lies in the tangent S_m, then the ratios $d^2u_1 : d^2u_2 : \cdots : d^2u_m$ can be so determined that (10) will lie in the line joining the points (8) and (9). Hence:

If through each point of a variety V_m in S_n there is a principal direction, then the coordinates must be particular solutions of the partial differential (12). *The osculating planes of curves enveloped by these principal directions will lie in the tangent S_m to the V_m.*

From equation (12) we see that the principal directions passing through each point of a hypersurface V_{n-1} in S_n form a quadric cone of $n-2$ dimensions, and the principal

directions through each point of a variety V_{n-k} $(2k<n)$ form a cone of order 2^k and dimensions $n-2k$. If $n=2m-1$ and $k=m-1$ the number of principal directions will be equal to 2^{m-1}. The curves enveloped by these directions correspond to the asymptotic curves on a surface in S_3 and in fact can be obtained by the consideration of conjugate directions just as in S_3. Take any two infinitely near points of V_m and draw the tangent spaces S_m to V_m at each of these points. The two S_m's will intersect in a line and this line is said to be conjugate to the direction defined by the two points of tangency. Using the same method as in a previous paper* by the author, the condition that the two directions du and δu be conjugate is found to be the vanishing of the $(m+2)$-rowed determinants of the matrix

$$\| x, x_1, x_2, \cdots, x_m, \sum_i \sum_j x_{ij} du_i \delta u_j \|.$$

The condition that the two conjugate directions should coincide is seen to be that the coordinates should satisfy equation (12). In this case then we have a new property of such curves, viz.: The tangent spaces S_m to V_m at two infinitely near points of the curve enveloped by the principal tangents intersect in the line joining the two points, that is, in a principal tangent. By an easy extension of the method used by Segre† it can be shown that this is a property of such curves in general. Or it may be seen that if this property does hold in general there is an S_{m+1} having two-point contact (i. e., tangent at two infinitely near points) with the variety V_m. Now the tangent at x is determined by

$$x, \quad x_1, \quad x_2, \quad x_3, \quad \cdots, \quad x_m \tag{13}$$

and the tangent at the point

$$\Sigma x_i du_i \tag{14}$$

is determined by

$$\Sigma x_i du_i, \quad \Sigma x_{1j} du_j, \quad \Sigma x_{2j} du_j, \quad \cdots, \quad \Sigma x_{mj} du_j. \tag{15}$$

It is seen at once that the tangent space (13) contains the point (14) and that the tangent space (15) contains the point

$$\Sigma x_{ij} du_i du_j \tag{11}$$

* *Annals of Mathematics*, vol. 13, p. 89.

† Loc. cit.

Now if the space (13) contains (11), the two tangent spaces will have the line defined by the direction $du_1 : du_2 : \cdots : du_m$ in common. This is exactly condition (12). Hence:

The curves enveloped by the principal directions on a variety whose coordinates satisfy (12) *are such that the tangent S_m's at two consecutive points intersect in the line joining the two points.*

3. Let us now find the condition that a variety V_3 have through each point a tangent with four point contact with V_3. The condition may be got by first finding the condition that curves exist on V_3 whose osculating S_3 at any point coincides with the tangent S_3 to V_3 at that point. This requires that the two points

$$\Sigma x_{ij} du_i du_j + \Sigma x_i d^2 u_i \tag{16}$$

$$\Sigma x_{ijk} du_i du_j du_k + 3\Sigma x_{ij} du_i d^2 u_j + \Sigma x_i d^3 u_i \tag{17}$$

should lie in the S_3 determined by the points

$$x, \quad x_1, \quad x_2, \quad x_3.$$

This is equivalent to the relations

$$A\Sigma x_{ij} du_i du_j + \sum_1^3 B_i x_i du_i + Cx = 0, \tag{12}$$

$$A'(\Sigma x_{ijk} du_i du_j du_k + \Sigma x_{ij} du_i d^2 u_j) + \Sigma B_i' x_i du_i + C'x = 0. \tag{18}$$

In relation (12) the du_i define the direction of the curve. If (12) is satisfied, then the determinants of the matrix

$$\| \Sigma x_{ij} du_i du_j, x, x_1, x_2, x_3 \|$$

must vanish. Now if we change du_i into $du_i + \frac{1}{2} d^2 u_i$, this relation must still be satisfied; this leads to

$$\| \Sigma x_{ij} du_i d^2 u_j, x, x_1, x_2, x_3 \| = 0,$$

which used in (18) reduces that relation to

$$A'\Sigma x_{ijk} du_i du_j du_k + \sum_1^3 B_i' x_i + C'x = 0. \tag{19}$$

Hence:

If a variety V_3 is covered with a family of curves whose osculating S_3 at each point coincides with the tangent S_3 to V_3 at that point, then the coordinates must be solutions of two simultaneous partial differential equations, one of parabolic type of

the second order and the other of the third order having coincident roots of its characteristic equation. The curves in question are the characteristics.

If equation (12) is satisfied, we saw that the tangents in the direction du_i have three-point contact with V_3. If in addition (19) is also satisfied, since (17) lies in the tangent S_3, the quantities d^3u_i can be so chosen that (17) will lie on the line joining x to $x + dx$. Hence:

When the coordinates of a spread V_3 satisfy the differential equations (12) *and* (19), *through each point of the spread passes a tangent having four-point contact with the spread.*

In the same manner it can be shown that the necessary and sufficient condition that a spread V_m have in each point a tangent with r-point contact is that the coordinates should satisfy the $r - 2$ equations

$$
\begin{aligned}
&A\Sigma \frac{\partial^{r-1}x}{\partial u_i \cdots \partial u_{r-1}}\, du_1 du_2 \cdots du_{r-1} + \Sigma B_i x_i + Cx = 0,\\
&\cdots\cdots\cdots

NOTE ON MIXED LINEAR INTEGRAL EQUATIONS.

BY DR. W. A. HURWITZ.

(Read before the American Mathematical Society, September 12, 1911.)

By a mixed integral equation is here understood an equation in which an unknown function may be involved under several integral signs operating over regions of different numbers of dimensions, and in which also the values of the unknown function at isolated given points may occur. The highest number of dimensions of any region of integration may be termed the order of the equation; an equation entirely free from integral signs may also be regarded as a mixed integral equation of order zero. An equation in which regions of integration of different numbers of dimensions do not occur may be called pure; equations of this sort constitute the type usually studied.

The object of the present paper is to give some indication of the extent to which mixed (non-pure) *linear* integral equations may be expected to follow certain laws which have seemed to govern linear equations in general (e. g., pure integral equations and systems of algebraic equations). It is found that the analogy is not complete; the principal result, however, follows the lines of the usual theories. It is stated, for the case of equations of the first order, in the following

THEOREM: *Let* $f(x)$, $k_1(x)$, $k_2(x)$, $\cdots$, $k_m(x)$ *be continuous fuctions of* x *in the interval*

$$R_1: \quad a \leqq x \leqq b,$$

and $K(x, y)$ *a continuous function of* x *and* y *in the region*

$$R_2: \quad a \leqq x \leqq b, \quad a \leqq y \leqq b;$$

let $x_1, x_2, \cdots, x_m$ *be distinct points of* R_1. *Then if the homogeneous mixed linear integral equation*

$$(1) \qquad u(x) = \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has no continuous solution in R_1, *other than* $u(x) = 0$, *the non-homogeneous mixed linear integral equation*

$$(2) \qquad u(x) = f(x) + \sum_{i=1}^{m} k_i(x)u(x_i) + \int_a^b K(x, s)u(s)ds$$

has one and only one continuous solution in R_1.

To prove this, let us form the two normal orthogonal sets of principal solutions for the kernel $K(x, y)$

$$\varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x); \quad \psi_1(x), \psi_2(x), \cdots, \psi_n(x);$$

these satisfy the following conditions:

$$\varphi_j(x) = \int_a^b K(x, s)\varphi_j(s)ds; \quad \psi_j(x) = \int_a^b \psi_j(s)K(s, x)ds \qquad (j = 1, 2, \cdots, n);$$

$$\int_a^b [\varphi_j(x)]^2 dx = \int_a^b [\psi_j(x)]^2 dx = 1; \quad \int_a^b \varphi_i(x)\varphi_j(x)dx = \int_a^b \psi_i(x)\psi_j(x)dx = 0 \quad (i \neq j;\ i, j = 1, 2, \cdots, n).$$

The equations

$$u_0(x) = f(x) - \sum_{j=1}^{n} \psi_j(x) \int_a^b \psi_j(s)f(s)ds + \int_a^b K(x, s)u_0(s)ds,$$

$$u_i(x) = k_i(x) - \sum_{j=1}^{n} \psi_j(x) \int_a^b \psi_j(s)k_i(s)ds + \int_a^b K(x, s)u_i(s)ds \qquad (i = 1, 2, \cdots m)$$

possess continuous solutions, since in each of them the known function outside the integral sign is orthogonal to $\psi_1(x)$, $\psi_2(x), \cdots, \psi_n(x)$. Form the following two systems of $m + n$ linear algebraic equations in $m + n$ unknowns $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$:

$$(\mathrm{I}_1) \quad A_p - \sum_{i=1}^{m} A_i u_i(x_p) - \sum_{j=1}^{n} B_j\varphi_j(x_p) = 0 \quad (p = 1, 2, \cdots, m),$$

$$(\mathrm{I}_2) \quad - \sum_{i=1}^{m} A_i \int_a^b \psi_q(s)k_i(s)ds = 0 \quad (q = 1, 2, \cdots, n),$$

$$(\mathrm{II}_1) \quad \mathrm{A}_p - \sum_{i=1}^{m} A_i u_i(x_p) - \sum_{j=1}^{n} B_j\varphi_j(x_p) = u_0(x_p) \qquad (p = 1, 2, \cdots, m)$$

$$(\mathrm{II}_2) \quad - \sum_{i=1}^{m} A_i \int_a^b \psi_q(s)k_i(s)ds = \int_a^b \psi_q(s)f(s)ds \qquad (q = 1, 2, \cdots, n).$$

Either system (I) possesses a set of solutions $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$ not all zero, or else system (II) possesses a set of solutions.*

Under the second of these alternatives, the continuous function

$$u(x) = u_0(x) + \sum_{i=1}^{m} A_i u_i(x) + \sum_{j=1}^{n} B_j \varphi_j(x),$$

defined in terms of the set of solutions of (II), is a solution of (2), as may be verified by direct substitution in (2), with the aid of (II). A second solution of (2) is impossible if (1) has no solutions not identically zero, since the difference of two solutions of (2) is evidently a solution of (1).

Under the first alternative, the continuous function

$$u(x) = \sum_{i=1}^{m} A_i u_i(x) + \sum_{j=1}^{n} B_j \varphi_j(x),$$

defined in terms of the set of solutions (not all zero) of (I), is similarly a solution of (1). Finally, this solution $u(x)$ is not identically zero; for if $u(x) = 0$, then in particular $u(x_p) = 0$ $(p = 1, 2, \cdots, m)$, so that (by substituting the values x_p for x in the definition of $u(x)$ and making use of (I_1))

$$A_p = 0 \qquad (p = 1, 2, \cdots, m);$$

hence

$$0 = u(x) = \sum_{j=1}^{n} B_j \varphi_j(x),$$

and in virtue of the normal orthogonality of $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$, $\varphi_n(x)$,

$$B_j = 0 \qquad (j = 1, 2, \cdots, n),$$

in contradiction to the fact that some one of $A_1, A_2, \cdots, A_m$; $B_1, B_2, \cdots, B_n$ is not zero.

The theorem stated above has thus been justified. With this theorem, however, the exactness of the analogy to the ordinary theories of linear equations seems to end. The more detailed study of the mixed linear integral equation is reserved for a later communication; an indication of the differences which present themselves is furnished by the

* The fact that these two states of affairs are not mutually exclusive is unessential.

following remark, which is stated without proof. The rôle of the adjoint homogeneous equation in the ordinary theories* is here taken by a pure homogeneous linear integral equation together with linear integral auxiliary conditions

$$(1') \qquad v(x) = \int_a^b v(s)K(s, x)ds; \quad \int_a^b v(x)k_i(x)dx = 0 \qquad (i = 1, 2, \cdots, m);$$

but the numbers of independent solutions of (1) and (1′) need not, as in the usual cases, be the same. For example, there can never be more (and will in general be fewer) than n linearly independent continuous solutions of (1′), while there may under certain conditions be as many as $m + n$ linearly independent continuous solutions of (1).

CORNELL UNIVERSITY,
December 22, 1911.

NOTE ON THE GRAPHICAL SOLUTIONS OF THE FUNDAMENTAL EQUATIONS IN THE SHORT METHODS OF DETERMINING ORBITS.

BY PROFESSOR A. O. LEUSCHNER AND MR. B. A. BERNSTEIN.

(Read before the San Francisco Section of the American Mathematical Society, October 28, 1911.)

THE chief difficulty in the determination of preliminary orbits of new comets or asteroids arises from the fact that the geocentric or heliocentric distance of the new body enters into the problem as an unknown quantity. In the older methods the heliocentric distance often is assumed to be unity as a first approximation in the determination of comet orbits. The average heliocentric distance of the known asteroids is generally chosen as a first approximation for new asteroids. The true distances are then evaluated by laborious approximations.

In the short methods proposed by Professor Leuschner it is possible to make a direct determination of the geocentric dis-

* E. g., for the pure linear integral equation, the equation with transposed kernel; for the system of linear algebraic equations, the system with transposed (or conjugate) matrix.

tance at the middle date of the three underlying observations to as many decimals as may seem desirable. This is accomplished by a graphical solution in the determination of parabolic or circular orbits and by means of a table of geocentric distances for general orbits.

For the parabola or circle the fundamental equation is of the form

$$f(z) = (z - p')^2 + q'^2 - \frac{h}{[(z-c)^2 + s^2]^{\frac{1}{2}}} = 0,$$

where the unknown z represents the ratio of the geocentric distance of the body to the geocentric distance of the sun and where all other quantities are known constants. The roots of the equation are given by the intersections of the parabola

$$y = (z - p')^2$$

with the asymptotic curve

$$y = \frac{h}{[(z-c)^2 + s^2]^{\frac{1}{2}}} - q'^2.$$

The distance between the axes of symmetry of the two curves is $c - p'$, reckoned from the axis of the parabola. The equations may be written

$$y = z'^2, \quad \eta = \frac{h}{(\zeta^2 + s^2)^{\frac{1}{2}}},$$

where $z' = z - p'$, $\zeta = z - c$, $\eta = y + q'^2$. The equation of the asymptotic curve may also be written

$$\zeta = s \tan \theta, \quad \eta = \frac{h}{s} \cos \theta.$$

In the astronomical problem only the positive real roots of $f(z) = 0$ need to be considered. There are either one or three positive real roots. The portions of the asymptotic curve near its intersections with the parabola may be conveniently plotted (Fig. 1) without the aid of a trigonometrical table, as follows: Draw a circle with radius h/s about the origin $\eta = 0$, $\zeta = 0$ as center and draw the line $\eta = s$. Draw any vector from the origin making an angle θ with $\zeta = 0$. Through the intersection M of vector and circle draw MM'

parallel to $\eta = 0$ and through the intersection N of vector and $\eta = s$ draw NN' parallel to $\zeta = 0$. Then for the intersection P of MM' and NN' we have

$$\zeta = s \tan \theta \quad \text{and} \quad \eta = \frac{h}{s} \cos \theta$$

and, therefore, P is a point on the asymptotic curve.

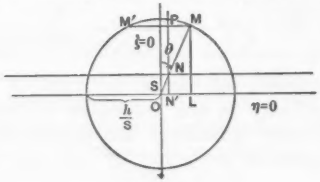

FIG. 1.

This construction was independently suggested by Mr. B. A. Bernstein. In practice the parabola $y = z'^2$ is first placed in proper position (with its vertex at $\zeta = p' - c$, $\eta = q'^2$ and its axis parallel to $\zeta = 0$) and a point P is located on either side of an intersection. The values of ζ corresponding to the intersections may then be read off and finally the corresponding values of z are given by $z = \zeta + c$. In locating a point P it is not necessary actually to draw either the vector or MM' or NN', but P may be located by means of rulers held in the proper position. In case of more than one solution the corresponding roots z may thus be obtained at one and the same time.

In the solution of an orbit without hypothesis regarding the eccentricity the equation is of the form

$$(z^2 - 2c + 1)^3(z - m)^2 - m^2 = 0.$$

There may be either one or two positive real roots which may be taken from a table with ψ and $1/m$ as arguments where $\psi = \cos^{-1} c$ and m are known constants. The table is an extension of Table XIIIa of von Oppolzer, Lehrbuch der Bahnbestimmung, volume 1.

In case of three parabolic solutions, the physical solution is determined by identification with a general solution. The mathematical parabolic solutions may thus be rejected in the course of computation. The only known cases with three parabolic solutions are comet Cruls of 1882 and

comet 1910*a*. Graphical solutions for these comets have been made by Mr. E. S. Haynes, John H. Pitman, and Miss S. Levy.

An application of the foregoing methods of solution for comet 1910*a* has been published by Professor Leuschner in the *Lick Observatory Bulletin*, No. 197. A full account with the tables is being printed in the *Publications of the Lick Observatory*, volume 7.

Mr. Bernstein gives a purely geometric solution of the fundamental equation that enters into the *general* solution of an orbit. He proceeds as follows:

From the equation

$$(1) \qquad (z^2 - 2c + 1)^3(z - m)^2 - m^2 = 0$$

it follows that

$$z - m = \frac{m}{(z^2 - 2c + 1)^{\frac{3}{2}}},$$

where only the positive sign before the radical need be considered.

Making the substitution $x = z - c$, and writing $1 - c^2 = s^2$ (since $c < 1$), the above becomes

$$(2) \qquad x + c - m = \frac{m}{(x^2 + s^2)^{\frac{3}{2}}}.$$

The roots of this equation are got from the intersections of the curve

$$(3) \qquad y = \frac{1}{(x^2 + s^2)^{\frac{3}{2}}}$$

with the straight line

$$(4) \qquad my = x + c - m.$$

If we make the substitution $x = s \cot \theta$, we get

$$(5) \qquad \begin{aligned} x &= s \cot \theta, \\ y &= \frac{1}{s^3} \sin^3 \theta = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta \end{aligned}$$

as the parametric equations of the curve represented by (3). This curve can now be constructed geometrically (Fig. 2) as follows:

Draw a line $\overline{A'A}$ parallel to the x-axis at a distance s above

it. With the origin O as center construct two circles having for radii $1/4s^3$ and $3/4s^3$ respectively. Let θ be any angle which a line $\overline{OQ}$ makes with $\overline{OX}$, the positive direction of the x-axis, and let $\overline{OQ}$ meet $\overline{A'A}$ in R and the circle of radius $3/4s^3$ in Q. Draw the angle $XOT = 3\theta$, and let the terminal

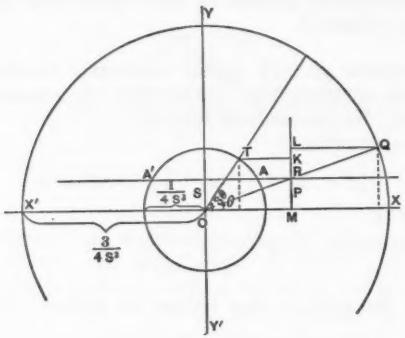

FIG. 2.

side of this angle meet the circle of radius $1/4s^3$ in $\underline{T}$. Draw $\overline{RM}$ perpendicular to $\overline{OX}$. From $\underline{T}$ and $\underline{Q}$ drop perpendiculars to $\overline{RM}$, meeting the latter in $\underline{K}$ and $\underline{L}$ respectively. On $\overline{MR}$ lay off $\overline{MP} = \overline{KL}$. Then P is a point on the required curve.

For

$$x = \overline{OM} = s \cot \theta$$

and

$$y = \overline{MP} = \overline{KL} = \overline{ML} - \overline{MK} = \frac{3}{4s^3} \sin \theta - \frac{1}{4s^3} \sin 3\theta$$

We thus obtain the geometric construction of the curve represented by equation (3). Its intersections with the line given by (4) are then easily found, and hence equation (2), and therefore (1), are solved.

ON A FUNCTIONAL EQUATION.

BY MR. A. R. SCHWEITZER.

(Read before the American Mathematical Society, October 28, 1911.)

1. ABEL,* in a discussion rigorized by Stäckel,† has shown (in effect) that if

$$f(x, u) = f(y, v) = f(z, w), \tag{1}$$

where

$$f(y, z) = u, \quad f(x, z) = v, \quad f(x, y) = w, \tag{2}$$

then there exists a function χ, dependent upon the function f, such that

$$\chi(f(x, y)) = \chi(x) + \chi(y).$$

In this paper it is desired to establish the following theorem:

If $f(v, w) = u$,‡ then there exists a function χ such that

$$\chi(f(x, y)) = \chi(x) - \chi(y).$$

In a sense, this theorem is a correlative of the theorem of Abel.

2. The method of proof of the preceding theorem resembles the procedure of Abel and Stäckel, l.c. Differentiating the relation

$$f(v, w) = u \tag{3}$$

with respect to y, z, x, we have

$$0 + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y} - \frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial z} + \quad 0 \quad - \frac{\partial u}{\partial z} = 0,$$

$$\frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} - \quad 0 \quad = 0.$$

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* *Crelle's Journal*, vol. 1 (1826), p. 11; cf. M. Cantor, *Zeitschrift für Mathematik und Physik*, vol. 41 (1896), p. 161.

† *Zeitschrift für Mathematik und Physik*, vol. 42 (1897), p. 323; cf. P. Hayashi, same journal, vol. 44 (1899), p. 346.

‡ This relation, with the preceding definitions of u, v, w, we call quasi-transitiveness. Abel, l.c., bases his discussion on the symmetry of $f(z, w)$ in x, y, z.

Therefore, on elimination of $\partial f/\partial v$, $\partial f/\partial w$,

$$\begin{vmatrix} 0 & \partial w/\partial y & \partial u/\partial y \\ \partial v/\partial z & 0 & \partial u/\partial z \\ \partial v/\partial x & \partial w/\partial x & 0 \end{vmatrix} = 0$$

or

$$\frac{\partial v}{\partial x} \cdot \frac{\partial w}{\partial y} \cdot \frac{\partial u}{\partial z} = -\frac{\partial v}{\partial z} \cdot \frac{\partial w}{\partial x} \cdot \frac{\partial u}{\partial y}. \tag{4}$$

The relation (4) is, of course, much more general than relation (3) from which it is derived, since (4) can be obtained, for instance, by differentiating any one of the relations*

$$\phi(v, w) = u, \quad \phi(w, u) = v, \quad \phi(v, u) = w.$$

Relation (4) differs, moreover, only in sign from a relation derived from (1) by Abel and Stäckel, and hence proceeding in the manner of the latter, we obtain from (4), if z = constant,

$$\frac{\partial w}{\partial y} \cdot \psi'(x) = -\frac{\partial w}{\partial x} \cdot \psi'(y) \tag{4'}$$

and as a solution† of (4'),

$$f(x, y) = \Omega(\psi(x) - \psi(y)),$$

where Ω remains to be further conditioned. Substituting in (3), we get

$$\Omega[\psi\Omega(\xi - \zeta) - \psi\Omega(\xi - \eta)] = \Omega(\eta - \zeta), \tag{5}$$

where

$$\psi(x) = \xi, \quad \psi(y) = \eta, \quad \psi(z) = \zeta.$$

On differentiating (5) with respect to y and z, and comparing the resulting relations, we find

$$\frac{\partial}{\partial \eta} \psi\Omega(\xi - \eta) = \frac{\partial}{\partial \zeta} \psi\Omega(\xi - \zeta). \tag{6}$$

Now we differentiate (6) with respect to y; then since

* It may perhaps be remarked in passing that $f(v, w) = u$ follows from $\phi(w, u) = v$ and $f(\phi(x, y), x) = y$.

† Cf. Abel, *Crelle's Journal*, vol. 2 (1827), p. 389. See also Mansion-Maser, Partielle Differentialgleichungen, Berlin (1892), pp. 34–37.

$$\frac{\partial}{\partial \eta}\psi\Omega(\xi-\eta) = -\frac{\partial}{\partial(\xi-\eta)}\psi\Omega(\xi-\eta)$$

we have

$$\frac{\partial^2}{\partial(\xi-\eta)^2}\psi\Omega(\xi-\eta)\cdot\frac{\partial\eta}{\partial y} = 0.$$

If $\partial\eta/\partial y = 0$, then $\psi(y) =$ constant, and hence $f(x, y) =$ constant, which we exclude; therefore

$$\frac{\partial^2}{\partial(\xi-\eta)^2}\psi\Omega(\xi-\eta) = 0$$

or

$$\psi\Omega(p) = cp + c_1, \quad p = \xi - \eta, \tag{7}$$

where c and c_1 are constants. Now let

$$\chi(x) = \psi(x) - c_1; \tag{8}$$

then

$$\chi(\Omega p) = cp,$$

and consequently,

$$\chi(f(x, y)) = c(\chi(x) - \chi(y)). \tag{9}$$

The constant c can be determined by substituting in (9) in accordance with (3); we find $c^2 - c = 0$, that is, since $f(xy) \neq$ constant, $c = 1$. Hence the theorem under § 1 is verified. Conversely, if

$$f(x, y) = \chi^{-1}(\chi(x) - \chi(y)), \tag{10}$$

then $f(x, y)$ has the property (3). If $f(x, y)$ is assigned, then by following Abel, l. c., volume 1, page 14, the function χ can be determined by the formula

$$\chi(x) = -\chi'(y)\cdot\int\frac{\partial w/\partial x}{\partial w/\partial y}dx, \tag{11}$$

where y is to be regarded as a constant. From this determination we deduce

1°. If $f(x, y) = \phi(x/y)$, then

$$\log c\cdot\phi(x/y) = \log cx - \log cy$$

and therefore, $\phi(x/y) = c'\cdot x/y$.

2°. If $f(x, y) = \phi(x - y)$, then $\phi(x - y) = x - y + c''$ where c' and c'' are constants.

In the case of the equation

$$f(v, u) = w \tag{12}$$

a discussion similar to the above is valid. It may be observed that relation (12) follows from (3) and (10), as does also a part of relation (1), viz.,

$$f(y, v) = f(z, w).$$

3. We conclude with a few remarks concerning the nature of the generalization of relation (3), which is perhaps less obvious than the corresponding generalization of relation (1) of Abel. Presumably the relation (3) occurs in a sequence of functional relations

$$f(f(t_1, x_2), f(t_1, x_1)) = f(x_1, x_2),$$

$$f(f(t_1, t_2, x_3), f(t_1, t_2, x_2), f(t_1, t_2, x_1)) = f(x_1, x_2, x_3),$$

$$f(f(t_1, t_2, t_3, x_4), f(t_1, t_2, t_3, x_3), f(t_1, t_2, t_3, x_2), f(t_1, t_2, t_3, x_1)) = f(x_1, x_2, x_3, x_4),$$

etc. This sequence leads to the sequence of linear partial differential equations of the first order

$$\frac{\partial u_2^{(0)}/\partial x_2}{\psi_2'(x_2)} + \frac{\partial u_2^{(0)}/\partial x_1}{\psi_2'(x_1)} = 0,$$

$$\frac{\partial u_3^{(0)}/\partial x_3}{\psi_3'(x_3)} + \frac{\partial u_3^{(0)}/\partial x_2}{\psi_3'(x_2)} + \frac{\partial u_3^{(0)}/\partial x_1}{\psi_3'(x_1)} = 0,$$

$$\frac{\partial u_4^{(0)}/\partial x_4}{\psi_4'(x_4)} + \frac{\partial u_4^{(0)}/\partial x_3}{\psi_4'(x_3)} + \frac{\partial u_4^{(0)}/\partial x_2}{\psi_4'(x_2)} + \frac{\partial u_4^{(0)}/\partial x_1}{\psi_4'(x_1)} = 0,$$

etc., where

$$f(x_1, x_2) = u_2^{(0)}, \quad f(x_1, x_2, x_3) = u_3^{(0)}, \quad f(x_1, x_2, x_3, x_4) = u_4^{(0)},$$

etc. The preceding equations have respectively the solutions,*

$$u_2^{(0)} = F_2(\psi_2(x_1) - \psi_2(x_2)),$$

$$u_3^{(0)} = F_3(\psi_3(x_1) - \psi_3(x_2), \quad \psi_3(x_1) - \psi_3(x_3)),$$

$$u_4^{(0)} = F_4(\psi_4(x_1) - \psi_4(x_2), \quad \psi_4(x_1) - \psi_4(x_3), \quad \psi_4(x_1) - \psi_4(x_4)),$$

etc., where F_2, F_3, F_4, etc., are arbitrary functions to be conditioned by the original functional relations.

CHICAGO, ILL.,
July, 1911.

* Mansion-Maser, l.c.

SHOP MATHEMATICS.

In the December issue of the BULLETIN there appeared a criticism of Holton's Shop Mathematics by Professor Charles N. Haskins which was in some respects so manifestly unjust as hardly to merit a reply. But since the criticism appears in a journal for which we have always had the highest regard, and since moreover there is an implied attack upon the ideals of secondary technical education, it may be well to show wherein this criticism falls wide of its intended mark.

The fundamental error under which Professor Haskins labors is the assumption, entirely unwarranted, that this book indicates the character of the mathematical instruction in technical high schools. As a matter of fact it was prepared mainly for the use of mechanics in the shop mathematics classes of an evening trade school, and finds simply an incidental, though very valuable, use in the (day) Technical High School, which is a school of full academic training, preparing students not only for industrial work directly but also for further study in the highest technical schools and colleges in the country. To meet the admission requirements of these institutions it is obviously necessary to give full courses in mathematics. And entirely aside from the preparatory courses no sensible teacher would fail to recognize the value of the purely mathematical element as a basis for application in the industries, as Professor Haskins points out at the close of his paper. This is not only assumed, but it is emphasized in all strong technical courses. The unfairness, or perhaps I should say the inaccuracy, of Professor Haskins's charge appears plainly in a sentence on page 138 in which he quotes, or misquotes, from the introduction a sentence which as there printed reads, "No attempt is made in this book to teach mathematical theory or principles." That is true. No such attempt is made *in this book;* for the purpose of the book does not require it. But Professor Haskins leaves out the phrase "in this book" and suggests that the author of the introduction was referring to the "character of the mathematical instruction in such schools." Professor Haskins therefore apparently misunderstood not only the kind of school for which the book was mainly intended,—viz., a trade school—but also the use which is actually made of it in the Technical High School—mistakes which really take away all the point in his attempt at serious criticism.

And yet there seems to be an element of unfairness in the general spirit of the criticism which should not go unchallenged. Professor Haskins assumes so much ignorance of practical affairs on the part of the author of this book, who is really a very practical man, and gives such a ridiculous turn to some of the problems selected for criticism as to provoke amazement, if not laughter, at the expense of the critic himself. For example, note the remarks about problem 17 on the raising of a building by means of jack screws. Now that was a problem based on facts. It was involved in the moving of one of our large public buildings within a stone's throw of the Springfield Technical High School. There were actually one hundred ¼ inch lead jack screws used; but it is by no means necessary to assume that there were one hundred men concerned in this operation,—one stationed at each jack screw. As a matter of fact there were five men, one for each of the five rows of jack screws, who operated each of the twenty jack screws in his row in order, giving a short turn at a time to each; and thus the building, which was a brick building, was successfully raised and moved. There is really no occasion for ridicule,—for comparing the force used to that exerted by bending the "little finger," or for comparing the bar with a "pen holder"; and it is inconceivable to the writer how anyone could find in this problem the necessary conception of one hundred laborers employed. And yet this ridiculous conception seems to have been suggested to Professor Haskins's mind by this by no means unusual case of lifting a brick building.

Turning now to the computations involved in this problem, it would hardly seem to be necessary to point out that this was not really given as a problem of computation that would occur in actual practice. It was expressly stated at the beginning that the element of friction was left out. It was therefore not intended to be a complete and actual problem at all. In fact, no contractor would need to figure out the length of these jack screw bars or the force required. For him and for most workmen such a matter is one of judgment acquired by experience. The problem was obviously chosen to illustrate the underlying *operating principles* of jack screws as *expressed in the formula;* and for this purpose there is certainly no harm in getting a computed result to thousandths of an inch. It is convenient to have such answers in order to test up the

accuracy of the arithmetical work of men in trade schools; for they need practice in computation quite as much as they need anything. But there is hardly one of these men who would not laugh down the critic who assumes that the author, or even they themselves, would be so lacking in common sense as to consider such a computed result as finding actual application in the practice of raising buildings. The interest of this problem for these men obviously consists in the insight it gives into the meaning of a certain mechanical formula. It is for the engineer, or for the workman who aspires to become an engineer, to make corrections for friction and figure out, if necessary, the value of the force that could actually be applied with a jack screw of known dimensions and determine the number required. But that is an entirely different matter.

The same reply may be given in general to the second criticism upon problem 12, page 51. Here again the main purpose is to illustrate the combination of several formulas; and the calculation is carried to its full extent in order that the work of the several members of the class may be checked up for accuracy. Friction is disregarded, the effort being to illustrate the mathematical basis on which practical work depends. Practical work itself would obviously involve many corrections for friction and other losses and imply approximate results.

Again in the criticism of the problem on spiral gears (page 45) Professor Haskins finds in this a good example of how *not* to apply mathematical formulas in the solution of such problems because, as he says, the method of solution given involves first multiplying by π and then dividing by π. Obviously this is unnecessary in a mathematical computation *to find one result*,—as, for example, the number of teeth. As a matter of fact, however, in this problem four results have to be found, viz., the circular pitch, the number of teeth, the blank diameter, and the lead of the spiral, all four of which results must be known in order to set up the machine to cut the gears. This is plainly indicated in the problem as given in the book under criticism in the words, "Calculate the other dimensions of the gears." Professor Haskins chooses to state the problem as one to determine the number of teeth (N) only,—a gross misstatement which, it would seem, could hardly have been made for any other purpose than to manufacture a point against the book. And yet we wish to be charitable. One

accustomed to think along the lines of pure mathematics only would not, of course, appreciate the real meaning of the mathematical layout of this problem; and it is to be presumed that this is Professor Haskins's real difficulty. He deprecates unnecessary mathematical work; and so do we. Short cuts in arithmetical computations are of course desirable if the workmen can be safely trusted to carry them out; but practical teachers find that it is not wise to dwell much upon short cuts in the elementary stages of the work. Since most of the men who use this book are far from being experts in computation, short cuts would be on the whole confusing, though of course when they are not confusing they are to be recommended. The kind of formalism Professor Haskins criticizes is not "excessive" and "pernicious" so long as it is necessary for a thorough understanding of the processes of computation.

CHARLES F. WARNER.

TECHNICAL HIGH SCHOOL,
SPRINGFIELD, MASS.

The above statements may be answered as follows:

(1) The review is said to contain "an implied attack on the ideals of secondary technical education." On the contrary those ideals, as stated by Dr. Woodward, were explicitly assumed as a standard by which the book was tested.

(2) With reference to the quotation on page 138 it would appear that since the sentence in question begins "Let us test *the book before us** by this standard, remembering that *its** object is etc.," the significance of the passage is the same whether the words "in this book" be omitted from the quotation or included therein.

(3) The reviewer hopes that the assumption that the book indicates the character of the instruction in secondary technical schools *is* unwarranted, but contends that a form of instruction which for the sake of convenience in testing the accuracy of the pupil's arithmetical work finds "no harm in getting a computed result to thousandths of an inch," when that result is the length of a jack-screw bar, effectually deprives the pupil of any real insight "into the meaning of a certain mathematical formula," and is pernicious formalism in that it does not compel the pupil to consider the significance and reasonableness of the numerical data and results.

* The italics were not in the review as printed. They seemed unnecessary.

(4) The jack-screw problem is said to be "a problem based on facts." Granted. Also "it was . . . not intended to be a complete and actual problem at all," and it is implied that one would be "lacking in common sense" "to consider such a computed result as finding actual application in the practice of raising buildings." Exactly. That was why it was chosen for criticism. The book aims to show the pupil "what the shop problems are."*

(5) The reviewer is accused of "gross misstatement" of the problem on spiral gearing and malicious intent on his part is implied. The reviewer would point out that of the four results above mentioned, one, the lead of the spiral, is not even mentioned in the solution of the problem as given by the author of the book. The other three quantities may be determined by the formulas: number of teeth $= N = Dp \cos Y$, blank diameter $= D + 2/p$, circular pitch $= \pi/p \cos Y$. Whether any single one or all three of these quantities are to be found from the given data, the determination does not involve operations equivalent to multiplication followed by division by π. This is true also of the determination of the dimensions of the driven gear, and of the numbers of the milling cutters to be used, both of which are given in the book, though not mentioned above. The question at issue is not one of "short cuts" in computation, but of comprehension of the principles and formulas underlying the computation. The reviewer sees no cause to retract any portion of his criticism of the problem.

CHARLES N. HASKINS.

SHORTER NOTICES.

Arithmétique générale. Par EMILE DUMONT. Paris, A. Hermann et Fils, 1911. 8vo. xvii+275 pp. 10 francs.

THIS volume presents a treatment of certain number fields and the ordinary laws of operation within these fields. The book is divided into four parts, of which the first treats of

* Introduction, p. viii.

natural numbers, the second of qualified numbers (nombres qualifiés), the third of complex numbers, and the fourth of quaternions and ternions. The treatment is elementary and no attempt is made to extend the subject or to present novel results. The preface, which by the way is one of the most interesting parts of this book, characterizes it quite fully and justly. We quote: "The definitions and the properties of numbers are generally studied by the young student at quite different times and from different classical works. . . . In all this there is no unity of method or point of view. . . . I have therefore attempted to furnish a treatment which should be of use to him who has finished his preliminary studies and who wishes to take a retrospective view of mathematical principles before he launches himself into the vast fields of mathematics and physics."

Mathematicians are classified as "logicians" and "rationalists." "In analysis the logicians reason on the written characters and in geometry they reason on words." "The rationalists consider mathematics as a preface to physics. They are constantly guided by reason based on experience."

The author follows throughout what he regards as the program of the rationalists. In his hands the fundamental proposition of mathematics (axioms, primitive propositions, or whatever we may call them) are corollaries of physics or of intuitional geometry. Numbers are used to express relations between "magnitudes." These "magnitudes" have properties which we obtain from direct experience with them and these properties we express numerically. The general properties of numbers are determined by the service which they are required to render. The notion of axioms and undefined symbols finds no place in this treatment. There in not an unproved proposition in the book. All the statements which we are wont to regard as assumptions appear under the caption of theorems. The proofs consist in references to remarks on the nature of magnitudes.

From the point of view of the rigorist this treatment is impossible. The reviewer must confess however that there is a certain charm about its naivété. It is refreshing to see this attempt to set mathematics directly and consciously into life and to validate its elementary propositions directly from a wealth of experience. When the true function of the rigorist is more generally understood we shall perhaps have more patience

with this sort of mathematics. With the rigorist the question at issue is frequently not whether the proposition which apparently is being considered is *true* or not, but whether it follows from some other proposition. That is, how much must we say to include by implication a certain other body of propositions.

It seems equally legitimate to inquire what must be the properties of systems of numbers in order that they shall express conveniently and accurately the varied phenomena that daily impinge upon us.

The prospective reader must judge for himself whether this is the sort of book he wants to read. Does he wish to study the modern development of the various algebras as based upon definite assumptions? Then this book is of no use to him. Does he wish to see an attempt to develop these algebras as corollaries of physics? Then it is probably the best book he could find.

N. J. LENNES.

Grundlagen der Geometrie. Von Dr. FRIEDRICH SCHUR. Leipzig, Teubner, 1909. vii + 192 pp.

THE Neuere Geometrie of Pasch marked the first effort to set up a complete set of fundamental statements for geometry —if we except the manifoldness development of Riemann. Following the appearance of that book there came a remarkable development of the subject that has thrown great light upon the logic of geometry. Schur wishes the present book to be considered as, in a sense, a revision of Pasch. Although the author expresses very strongly his indebtedness to Pasch, in only the most general way can the book be said to be such a revision.

In the general trend of his development the author follows Peano, in that congruence is obtained from motion or from projective geometry rather than directly from postulates, as is done by Hilbert, for example.

The general problem is formulated as follows: "To set up a simple and complete system of intuitive facts or axioms, entirely independent of one another, from which geometry can be derived by purely logical processes. To deserve the name geometry, axioms must be employed which express the results of the simplest and most elementary consideration of geometric figures, from which they are obtained by abstraction." This of course bars out the idea of space as a number manifold.

Although in a general way the effort is made to have the axioms independent, there would seem to be no effort to have each axiom contain but a single statement. In this respect the book lacks the elegance of logical form of the work of Veblen.

In § 1, the "projective postulates" are developed. The point and the segment (Strecke) are employed as undefined symbols, and the postulates concerning them in this chapter are sufficient to develop the ordinary connective properties of space and the order relations of points, and to characterize the space as being of exactly three dimensions. These axioms are equivalent to the first two groups of axioms of Hilbert. The line, plane, and space are formally defined in terms of point and segment, and the separation theorems are proved.

In § 2, after proving Desargues's theorem concerning perspective 3-edges in a bundle of lines in 3-space, it is shown that a line and a plane not containing it, or two lines in the same plane, determine a set of lines having the properties of a bundle. If the existence of the point of intersection is not certain, the bundle thus obtained is said to determine an ideal point. Similarly ideal lines are defined by means of non-intersecting planes. Desargues's theorem is then proved in the plane, and the complete duality of the system of actual and ideal points is shown. That the Desargues theorem can not be proved in the plane alone is shown by means of the non-desarguesian geometry of Moulton. Finally an ideal plane is defined as the locus of lines which join an ideal point to all points of an ideal line not containing it, and the ideal connection properties of 3-space are then complete. Since the only assumption concerning the intersection of lines in a plane is that regarding the points of a triangle, i. e., a restricted domain, the development can serve as a basis for the euclidean and non-euclidean geometries which differ regarding intersection properties of two lines in the unlimited plane. The central collineation and collinear reflections are then defined, and used for the definition of harmonic points.

In § 3, postulates are introduced to make precise the concept of motion, and on this are based definitions of absolute pole and polar, perpendicular lines and planes, folding, and rotation, and the proof of Pascal's theorem for the degenerate conic. Finally congruence of segments is defined as follows: Two segments are called congruent, if one can be transformed into the other by a motion.

In § 4, the fundamental theorem of projective geometry is proved and the algebra of segments and resulting analytic geometry are developed. The fundamental theorem is proved by making use of the postulates of motion, a procedure that appears less natural from the projective point of view than it would be to choose as a postulate either this theorem itself, or the theorem of Pascal.

The projective development is then applied for the derivation of the fundamental metric forms of non-euclidean geometry. The equations of lines are obtained, and the trigonometric functions sine and cosine are defined projectively. The theory of triangles then follows readily. The characteristic constant k occurring in these formulas depends on the character of the absolute involution obtained on each line. Congruence of triangles is developed, and construction of the triangle from given parts. The proof of the independence of the parallel axiom is made by exhibiting in the classical way the analytic number spaces for which each hypothesis concerning parallels is valid. The relation of the parallel axiom to the angle-sum of a triangle and to motion is also discussed. In § 7, plane geometry is developed without the use of 3-space, following the methods of Hjelmslev. In § 8, the role of the Archimedean axiom is treated. The proof that while the Pascal theorem does not follow from the projective postulates alone, it does follow from these with the addition of the Archimedean postulate, is especially interesting.

As a whole the book is a valuable addition to the literature of the subject. More references to original sources would add much to its value. It is not free from typographical errors, but those noted are fairly evident from the context.

F. W. OWENS.

Précis de Mécanique rationnelle. Introduction à l'Étude de la Physique et de la Mécanique appliquée. Par P. APPELL et S. DAUTHEVILLE. Paris, Gauthier-Villars, 1910. 8vo. 729 pp. 25 francs.

To condense an enormous Traité in three volumes and over 1,800 pages into a *précis*—a very significant term—of one volume and about 700 pages, demands a treatment which is more than a mere deletion of certain parts. The whole must to some extent be recast. Everyone has seen abridged treatises which were worthless and by the side of the original quite ob-

scure. In the present case however the task has been successfully done, and we have in this short course an admirable treatise on mechanics which covers a very wide range in a thorough manner. As the title indicates, the subject considered is rational mechanics, yet it is brought so close to applied mechanics that the passage from one to the other is very easy. Although the standpoint of the text is modern, yet it is above all a book for the undergraduate student.

The object of mechanics is stated to be the solution of two problems:

To find the motion of a system of bodies under the action of given forces;

To find the forces that will produce a given state of motion. By force, it is carefully explained, is meant a purely fictitious cause, not a real physical cause. It is merely an abbreviation for the product of acceleration by mass. Mass is defined to be the arbitrary number that may be assigned to a material point, such that the inverse ratio of two such arbitrary numbers is equal to the ratio of the accelerations they produce in each other. Thus all problems are reduced to questions of acceleration, that is, to questions of functions of four variables x, y, z, t and the derivatives of the first three as to the fourth. In the preface of the second edition of the third volume of the Traité, attention is called to a note at the end of the volume on "la théorie de l'action euclidienne." The object of this note is to reduce the whole theory of mechanics to the consideration of certain invariants in euclidean displacements. The note is not repeated in the Précis, but the spirit of it is in evidence throughout. The development of the course, however, does not demand on the part of the student a familiarity with group theory nor even great knowledge of analysis. Some acquaintance with differential equations is necessary. Intuitive methods are used to some extent and particular problems are developed in detail usually, in order to make the use of general theorems quite clear. The ripe experience of the authors shows itself continually, so that one feels that he has here a finished product of many years preparation. We will undertake to indicate some of the main features in some detail.

There are three parts: Preliminary Notions, Statics, and Dynamics. In the first we find chapters on vectors, kinematics, and the principles of mechanics; in the second, on equilibrium of points, of solids, of deformable systems; in the third, on

dynamics of a point, moments of inertia, the seven universal equations of motion, motion of a solid, friction, impact, virtual work, d'Alembert's principle and Lagrange's equations, attraction and potential, hydrostatics, and hydrodynamics. A selection of examination problems closes the book. The omissions are in general of the more elaborate problems or the more difficult parts of the theory.

Although, as seems to be the more common custom in France, the notation of vectors is not used in this course, nevertheless the first chapter is devoted to a sketch of the different kinds of vectors and their more important theorems. Three classes are noted: the free vectors, the sliding (glissant) vectors, and the bound vectors. The terminology of these is freely used throughout the course. The first class include those which combine according to the parallelogram law, such as translations, velocities, accelerations, axes of couples, and like quantities. The second class comprise such quantities as forces acting on a rigid body, linear moments, moments of momentum, screws, etc. The third class are the localized vectors, such as a velocity field, acceleration field, electromagnetic field, vortex field. The distinction between axial and polar vectors is also explained. Two invariants are noted: $X^2+Y^2+Z^2$ and $LX+MY+NZ$. The first is the square of the length of the resultant, the second represents the sum of the volumes of all the tetrahedra made by taking the sliding vectors in pairs as the opposite edges of tetrahedra.

The chapter on kinematics has been doubled in length over that in the Traité. This is partly from the examples introduced, but it is also due to the very much more detailed treatment of the definitions and the theorems. There is a distinct gain for the elementary student. The development produces the usual expressions for the velocity and the acceleration in various coordinates, discusses the motion of a solid, and the problems of relative motion. If one were to offer a suggestion on this part of the book, it would be that a brief consideration of relativity in general might not be amiss.

The chapter on the principles of mechanics lays down the following as the foundation of what follows:

1. Inertia: every material point which is supposed to be isolated has no acceleration.

2. Two material points moving with any velocities produce in each other oppositely directed accelerations along the line

joining the two points. (The action may be due to electricity, gravity, magnetism, etc.)

3. The ratio of the numerical values of these accelerations is constant, that is, the ratio is independent of the physical conditions of the points, or of their motion. These ratios are inversely as the ratios of the masses, one mass being purely arbitrary. Masses are analogous to chemical equivalents.

4. The accelerations of a point due to several points compound according to the parallelogram law.

One might suggest that here a brief note or paragraph could have been given on the definitions of mass which depend on the velocity, and the place of the "new mechanics" with regard to the "Newtonian mechanics." Such an explanation would orient the student as to some of the more recent work in mechanics.

The notion of force function and field of force are introduced in connection with the idea of work. The principal fields are mentioned: uniform field, gravity field, electric field, magnetic field.

The general conditions of equilibrium are stated in the succinct form: That any system be in equilibrium it is necessary and sufficient that the forces form a system of sliding vectors equivalent to zero. In the discussion of the equilibrium of a solid body we miss the single condition given in the Traité: for the equilibrium of a solid body the necessary and sufficient condition is that the sum of the moments of all the forces with respect to each of the edges of any tetrahedron vanish for each edge. The simplicity of this vector statement can scarcely be improved. It is also very general, as the tetrahedron may be chosen to suit the investigation. The discussion of the subject of the chapter is more detailed than in the Traité, with an increase of clearness and simplicity. The consideration of virtual work has been transferred to a later chapter.

Chapter VII, opening the third part, is devoted to the dynamics of a point. These topics are discussed: general theorems, straight motion, curvilinear motion of a heavy point in a vacuum and in a resisting medium, electrified particle in a field, motion of a point on a surface, equilibrium and relative motion. The text follows the Traité, the more special or complicated cases being omitted. The illustrative examples are carefully worked out and discussed.

The chapters on Lagrange's equations, d'Alembert's prin-

ciple, and canonical equation have been placed farther on, after the treatment of virtual work.

Chapter IX considers the dynamics of systems, general theorems, and the seven universal equations of motion. These theorems include the conservation of the motion of the center of gravity of the system, of the sum of the moments of momentum, and of vis viva. These lead to seven general equations of motion, three for the center of gravity, three for the moments of momentum, one for the kinetic energy. The internal forces do not enter the first six, but are present in the last. It is shown that these equations remain unchanged if the coordinates are referred to axes moving parallel to themselves with a uniform motion, or to axes moving parallel to themselves and always having the center of gravity as origin. It is pointed out that the motion in any case is reducible to the motion of the center of gravity plus that of the parts of the system with reference to the center of gravity. Thus the question of relative and absolute motion enters again. The meaning of energy and the law of conservation of energy close the chapter.

Chapter X takes up the motion of a solid body. The discussion is divided into the study of motion about an axis, motion parallel to a plane, screw motion, motion about a fixed point, motion about a point in a field which is directed to the point, motion about a fixed point in a gravity field, and motion of a free solid. The particular problems worked out are the composite pendulum, a homogeneous straight wire passing through a fixed point and sliding in a plane, a heavy plate sliding on a line in a plane. The Traité considers several other problems, some of which might have come in here: the Atwood machine, wheel rolling down an incline, double cone, the hoop, the top.

Friction is considered in the following chapter. It is practically defined as the effect of a force tangential to the surface, and two couples, one whose axis is the normal, and one whose axis is tangent. There are thus friction of sliding, of spinning, and of rolling. The spinning friction is omitted from the discussion.

Chapter XIII is devoted to developing the principle of virtual work which gives rise to what is here rather technically called analytic mechanics. From this principle are deduced the general conditions of equilibrium, and its importance is pointed out for that treatment of dynamics which reduces the problems

to problems in statics. The next chapter takes up the principle of d'Alembert, which is made the sole basis of the development. The principles of Gauss and of Hamilton and the principle of least action, are given in the Traité but omitted here. Lagrange's equations, the distinction between holonome and non-holonome systems, small movements about a position of equilibrium, and the canonical equations are discussed. These are all illustrated with examples worked out in full and with practical directions as to the use of the theorems. It is also suggested that many of the problems could as well have been solved by methods previously given. The next chapter completes the treatment of an earlier one on impact. A method analogous to the principle of d'Alembert is used.

In Chapter XVI we find Green's theorem, stated in the usual way and also in the vector form: the integral of the flux of the field through a closed surface is equal to the integral of the divergence of the field over the points enclosed by the surface. This theorem is then applied to the theory of attraction and potential.

The last chapters are concerned with the equilibrium of fluids, and with hydrodynamics. Both Lagrange's and Euler's methods are developed, and there is a brief treatment of vortices.

Taken as a whole, it is difficult to see how the text could be better. The book has a finished character, and is so practical as a text that one lays it down well pleased with its form and contents.

JAMES BYRNIE SHAW.

Systèmes cinématiques. Par L. CRELIER. Paris, Gauthier-Villars, 1911, 99 pp.

THIS monograph is one of the recent numbers of the well-known French collection Scientia. The author confines himself to the solution of a number of problems in generating plane algebraic curves by link motions, so that the general title of kinematic systems for the ground covered, even when limited to the plane, seems not entirely justified.

We should expect an introductory chapter on the general principles of link motions and the transformations realized by them. The theorems of Kempe and Koenigs concerning the realization of all algebraic curves and surfaces by linkages should at least be mentioned. A treatment of general propositions like these would be of far greater importance

than the solution of any number of particular problems without correlation.

Among the list of references we miss the names of Burmester, Somoff, and in view of the subject matter treated, De Ross, who in the *Revue Universelle des Mines* published a series of articles on linkages, their different forms and uses. A translation of these articles appeared 1879 in Van Nostrand's Science Series. As in Crelier's kinematic systems, the tracing of curves is the main object of De Ross's investigations. While De Ross uses Peaucellier's inversor and its modifications as a generator, Crelier does not mention it at all.

The first three chapters deal with the curves which are produced by points and lines in connection with the motion of a right angle subject to certain conditions. In the next chapter conchoidal circular systems and curves are studied. The displacement of the extremities of a straight line of constant length along two rectangular lines, and on a straight line and a circle are the subjects of the concluding two chapters.

A number of new curves are obtained from the various motions. The treatment is throughout clear and simple and does not require more than the elements of plane analytic geometry and calculus. In some parts the typography is poor and a number of figures lack precision and neatness of execution.

ARNOLD EMCH.

NOTES.

AT the meeting of the London mathematical society held on January 11 the following papers were read: By W. H. Young, "Successions of integrals and Fourier series"; by G. H. Hardy and J. E. Littlewood, "A new condition for the truth of the converse of Abel's theorem"; by A. Cunningham, "On Mersenne's numbers."

AT the meeting of the Edinburgh mathematical society on February 8 the following papers were read: By Professor Goldziher, "On graphical integration"; by Dr. Muirhead, "A mechanism for solving equations of the nth degree"; by G. Philip, "The geometry of the general pedal curve"; by W. Gentle, "An extension of the remainder theorem."

AMONG the prizes to be awarded by the Paris academy of sciences in 1913, the following are for achievements in pure and applied mathematics: The Francoeur prize (1000 fr.) for discoveries or works useful to the progress of pure or applied mathematics; the Bordin prize (3000 fr.) for improving in some important point the arithmetic theory of non-quadratic forms; the Poncelet prize (2000 fr.) for work in applied mathematics; the Lalande prize (540 fr.) in theoretical astronomy; the Pontécoulant prize (700 fr.) in theoretical astronomy; the Saintour prize (3000 fr.) for the most meritorious memoir in mathematics; the Petit d'Ormay prize (10000 fr.) in pure and applied mathematics; the Pierson-Perrin prize (5000 fr.) in mechanics and physics; the Wilde prizes (one of 4000 fr., one of 2000 fr.) for the most important contributions to science made during the two years immediately preceding the time the awards are made. The competition for all these prizes is unrestricted.

The academy has awarded the Saintour prize (3000 fr.) for 1911 to Professor J. DRACH, of the University of Toulouse, for his memoir on "Groups of rationality of differential equations."

THE international commission on the teaching of mathematics announces the following programme for its next session, in connection with the fifth congress of mathematicians at Cambridge: At the general session the president, Professor F. KLEIN, will give a summary of the results already reached, and formally request that the commission be continued until the next congress in 1916. The commission will then hold three sessions in affiliation with section IV of the congress. At the first the reports of the chairmen of the national subcommissions will be received; at the second, the report of committee A of the Milan meeting will be discussed, and at the third, the report of committee B. (See BULLETIN, volume 18, page 36.) Thus far, forty-nine monographs have been published by the various sub-committees, and a number of others will appear during the next few weeks.

THE report of the national committee of fifteen on a geometry syllabus, which has been under consideration for nearly three years, has finally been published in a pamphlet of 70 pages and is ready for distribution to teachers of geometry,

and all others interested. This report was prepared under the joint auspices of the American Federation of Teachers of the Mathematical and Natural Sciences and the National Education Association. It includes a historical introduction and sections on axioms and definitions, on exercises and problems, and the syllabus itself including both plane and solid geometry. It is the hope of the committee that this report may be of great service to all teachers of geometry, and to this end that it may have a wide distribution among all interested. Copies may be secured gratis upon application to the Commissioner of Education, Department of the Interior, Washington, D. C.

THE firm of Eimer and Amend, 205–211 Third Avenue, New York City, have just issued an Illustrierter Spezialkatalog mathematischer Modelle und Apparate für den Unterricht in der Planimetrie, Stereometrie, Trigonometrie und verwandten Fächern, entworfen von G. Koepp und anderen bewährten Fachmännern. The catalogue contains besides an index, 128 pages of descriptions and prices of several hundred models, largely for instruction in the secondary schools, and 324 illustrations.

MR. W. M. COATES, fellow of Queen's College, Cambridge, died at Cambridge January 16, at the age of 54 years. Mr. Coates had the reputation of being the most successful mathematical coach since the time of the late Dr. E. J. Routh, under whom he won the third wranglership in 1883.

RECENT catalogues of new and second-hand books:—G. J. Göschen'sche Verlagshandlung, Marienstrasse 18, Leipzig, publications of this firm in mathematics and allied sciences, 1907–1911.—A. Hermann et Fils, 6 rue de la Sorbonne, Paris, 300 titles in mathematics and exact sciences.—E. Beyer, Schottengasse 7, Vienna, catalogue 61, 1296 titles in mathematics, astronomy and meteorology.—G. Fock, Schlossgasse 7–9, Leipzig, catalogue 391, 3500 titles in mathematics and physics, also catalogue 407, 4000 titles in mathematics, physics and astronomy.—J. Schweitzer, Lenbachplatz 1, Munich, catalogue 54, 2300 titles in insurance.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

D'ARCAIS (F.). Analisi infinitesimale. 3a edizione. Volume 1. Fascicolo 1. Padova, 1911. 8vo. 448 pp. L. 12.00

BACHMANN (P.). See KLEIN (F.).

BAUM (K. T.). Der Kreis und seine Quadrate, die arithmetische Wage, $\pi = 3.125$ oder $3\frac{1}{8}$ und die Quadratur des Kreises. Saarbrücken, 1911. 8vo. M. 1.20

BÖHM (K.). Axiome der Arithmetik. Heidelberg, 1911. 8vo. 11 pp.

BRENDEL (M.). See KLEIN (F.).

BÜCHER, Neue, über Naturwissenschaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels, nach Wissenschaften geordnet.) Mitgeteilt Herbst 1911. Leipzig, Hinrichs, 1911. 8vo. 68 pp. M. 0.30

CAIN (W.). A brief course in the calculus. 3rd edition, revised. New York, Van Nostrand. 8vo. 11+281 pp. $1.75

COFFIN (J. G.). Vector analysis. An introduction to vector-methods and their various applications to physics and mathematics. 2nd edition. New York, Wiley, 1911. 12mo. 22+262 pp. Cloth. $2.50

COHEN (A.). An introduction to the Lie theory of one parameter groups. New York, Heath, 1911. 12mo. 247 pp. $2.00

DINI (U.). Sugli sviluppi in serie per la rappresentazione analitica delle funzioni di una variabile reale date arbitrariamente in un certo intervallo. Pisa, 1911. 8vo. 480 pp. L. 12.00

DIXON (A. C.). On the series of Sturm and Liouville, as derived from a pair of fundamental integral equations instead of a differential equation. London, Dulau, 1912. 4to. 22 pp. 1s.

EGOROFF (C.). Le théorème de Fermat. St. Petersburg, 1911. 8vo. 18 pp. M. 3.00

EHRENBERG (H.). Kants mathematische Grundsätze der reinen Naturwissenschaften. Heidelberg, 1910. 8vo. 60 pp.

FAGNANO (G. C.). Opere Matematiche. Milano, 1911. 8vo. L. 45.00

GAMBIER (G.). Le mathematicien François Viète. Généalogie de sa famille. La Rochelle, 1911. 8vo. 31 pp.

HAEUSSLER (J. W.). Allgemeiner Beweis des Fermatschen Satzes. Berlin, Krayn, 1912. 8vo. 20 pp. M. 0.80

HEIDELBERG (P.). Allgemeiner Beweis des Fermatschen Satzes. 3te Auflage. Coblenz, Erben, 1911. 8vo. 3 pp. M. 0.60

HEFFTER (L.). Über eine vierdimensionale Welt. Freiburg, 1911. 8vo. M. 0.90

HOLST (E.). See MATEMATIKKEN.

KARNASH (D.). Beweis für den Fermatschen Satz. Berlin, Mayer und Müller, 1911. 8vo. 17 pp. M. 1.20

KATALOG mathematischer Modelle für den höheren mathematischen Unterricht. Leipzig, Schilling, 1911. 8vo. 14+172 pp. M. 1.20

KELVIN (Lord). Mathematical and physical papers. Vol. 6. London, Cambridge University Press, 1912. 8vo. 386 pp. 10s.

KIEPERT (L.). Grundriss der Differential- und Integral-Rechnung. 2 Teile. 1: Differential-Rechnung. Hannover, Helwing, 1912. 8vo. 20+863 pp. M. 13.50

——. Grundriss der Differential- und Integral-Rechnung. 2 Teile. 2: Integral-Rechnung. Hannover, Helwing, 1912. 8vo. 23+738 pp. M. 13.50

KLEIN (F.). und BRENDEL (M.). Materialen für eine wissenschaftliche Biographie von Gauss. Heft 1: Über Gauss' zahlentheoretische Arbeiten (von P. Bachman). Leipzig, Teubner, 1911. 8vo. 3+54 pp. M. 2.00

KOCH (W.). Beiträge zur affinen Geometrie der Flächen zweiten Grades. Kiel, 1910. 8vo. 54 pp.

KRAUSE (M.). Theorie der elliptischen Funktionen. Leipzig, 1912. 8vo. 6+186 pp. M. 3.60

LEVY (P.). Sur les équations intégro-différentielles définissant des fonctions de lignes. (Thèse.) Paris, Gauthier-Villars, 1911. 4to. 121 pp.

LIETZMANN (W.). Der pythagoreische Lehrsatz, mit einem Ausblick auf das Fermatsche Problem. Leipzig, Teubner, 1911. 72 pp. M. 0.80

LÖFFLER (E.). Die Ziffern und Ziffernsysteme der Kulturvölker in alter und neuer Zeit. Leipzig, Teubner, 1911. 84 pp. M. 0.80

MASSENET (G.). Eléments de calcul infinitésimal. Paris, Challamel, 1912. 8vo. 8+292 pp.

MASURKEWITZ (J.). Über die Grenzen der Lösbarkeit der Gleichung $x^\lambda + y^\lambda = z^\lambda$ in ganzen Zahlen. (Grosser Fermatscher Satz.) Berlin, 1911. 8vo. 4 pp. M. 0.50

MATEMATIKKEN og Naturvidenskapen ved det K. Fredriks Universitet, 1811–1911. Särtryk av Festskrift. Matematikken av E. Holst. Kristiania, 1911. 4to. 151 pp. M. 6.50

MILNE (W. P.). Projective geometry. New York, Macmillan, 1911. 12mo. 15+148 pp. $0.75

NIEWENGLOWSKI (B.). Cours de géométrie analytique. Tome 2: Construction des courbes planes. Compléments relatifs aux coniques. Courbes définies par les équations différentielles. 2e édition. Paris, Gauthier-Villars, 1912. 8vo. 4+328 pp. Fr. 9.00

PAOLETTI (G.). Applicazioni della teoria di Lie sui gruppi continui di trasformazioni alla risoluzione delle equazioni differenziali. Rocca S. Casciano, 1911. 8vo. 64 pp. L. 2.25

PUZYNA (J.). Über Systeme von Kurven mit der Gruppe pseudolinearer Substitutionen. (Polish.) Krakau, 1911. 8vo. 46 pp.

SCHUBERT (H.). Niedere Analysis. (Sammlung Schubert. Band 45.) 2ter Teil: Funktionen, Reihen, Gleichungen. 2te durchgesehene Auflage. Leipzig, Göschen, 1911. 8vo. 5+215 pp. Cloth. M. 3.80

STERZINGER (O.). Zur Logik und Naturphilosophie der Wahrscheinlichkeitslehre. Leipzig, 1911. 8vo. M. 4.50

STOFFAES (E.). Cours de mathématiques supérieures à l'usage des candidats à la licence ès sciences physiques. Premier tome: Compléments d'algèbre élémentaire. Dérivées. Equations. Géométrie analytique. Différentielles et intégrales. Paris, Gauthier-Villars, 1911. 8vo. 6+398 pp. Fr. 10.00

SUINI (A.). Delle definizioni di retta e di piano quale vere basi della geometria: studio di filosofia matematica, a complemento dell'altro contenuto nell'opuscolo dal titolo: La confutazione della geometria non euclidea e la teoria naturale delle parallele. Piacenza, Porta, 1911. 8vo. 18 pp. L. 1.00

TURRIÈRE (E.). Sur les congruences de normales qui appartiennent à un complexe donné. (Thèse.) Toulouse, Privat, 1911. 4to. 88 pp.

WIELEITNER (H.). Geschichte der Mathematik. (Sammlung Schubert. Band 63.) 2ter Teil: Von Cartesius bis zur Wende des 18ten Jahrhunderts. 1te Hälfte: Arithmetik, Algebra, Analysis. Bearbeitet unter Benutzung des Nachlasses von A. v. Braunmühl. Leipzig, Göschen, 1911. 8vo. 8+251 pp. Cloth. M. 6.50

II. ELEMENTARY MATHEMATICS.

ADLER (A.) und MÜLLER (E.). Berichte über den mathematischen Unterricht in Österreich. Band 9. 1ter Teil: Der Unterricht in der darstellenden Geometrie an den Realschulen und Realgymnasien (von A. Adler). 2ter Teil: Der Unterricht in der darstellenden Geometrie an den technischen Hochschulen (von E. Müller). Wien, Hölder, 1911. 8vo. 124 pp. M. 2.40

BOUVART (C.) et RATINET (A.). Règles et formules usuelles servant de supplément aux tables de logarithmes, contenant: 1, une liste des nombres premiers et des nombres n'ayant pas de diviseur moindre que 13; 2, les racines carrées des nombres de 1.0 à 9.9 et de 10 à 100; 3, les longeurs des arcs et des lignes trigonométriques dans les deux systèmes de division; 4, les règles et les formules usuelles des cours de mathématiques et de physique; 5, des tableaux relatifs aux poids atomiques, aux densités et aux températures de fusion et d'ébullition. 3e édition, revue. Paris, Hachette, 1912. 8vo. 47 pp. Fr. 0.60

BRANDENBERGER (K.). Der mathematische Unterricht an den schweizerischen Gymnasien und Realschulen. Der mathematische Unterricht in der Schweiz. Band 4. Basel, Georg, 1911. 8vo. 167 pp. M. 3.50

CHINI (M.). Lezioni di algebra elementare. Seconda edizione del I volume delle Lezioni di algebra, ad uso dei licei. Livorno, Giusti, 1912. 8vo. 8+171 pp. L. 1.80

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GAJDECZKA (J.). Lehrbuch der Geometrie für die oberen Klassen der Mittelschulen. 2te Auflage. Nach den Lehrplänen von 1909 umgearbeitet von E. Kaller. Wien, Deuticke, 1911. 8vo. 5+267 pp. Cloth. M. 3.60

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GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Zum Gebrauche für Schule und Praxis bearbeitet. Stuttgart, Wittwer, 1912. 8vo. 31+177 pp. Cloth. M. 2.50

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GROSSMANN (M.). Der mathematische Unterricht an der eidgenössischen technischen Hochschule. (Der mathematische Unterricht in der Schweiz. Band 7.) Basel, Georg, 1911. 8vo. 52 pp. M. 2.00

HALL (A. G.) and FRINK (F. G.). Plane and spherical trigonometry. New York, Holt, 1911. 8vo. 10+176 pp. $1.00

HALL (W.). Tables and constants to four places for use in technical and nautical computation. New York, Putnam, 1911. 8vo. 9+60 pp. $0.90

HARTL (H.). Übungsbuch für den Unterricht in der allgemeinen Arithmetik und Algebra an Werkmeisterschulen, Baugewerkschulen und verwandten Lehranstalten. Wien, Deuticke, 1911. 8vo. 6+163 pp. Cloth. M. 2.40

HOBSON (E. W.) A treatise on plane trigonometry. 3rd edition. London, Cambridge University Press, 1912. 8vo. 400 pp. 12s.

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